Background for the Teacher

Making Sense of Division with Fractions

> *"Division by fractions, the most complicated operation with the most complex numbers, can be considered as a topic at the summit of arithmetic."*
>
> Liping Ma (1999)

> *"Division of fractions is often considered the most mechanical and least understood topic in elementary school."*
>
> Dina Tirosh (2000)

Division of fractions often evokes anxiety in adults. Many recall a process of inverting and multiplying but very few understand why that procedure works. By providing a three-year period—Grades 5, 6, and 7—for students to learn to multiply and divide with fractions, the authors of the Common Core State Standards aim to help generations of learners understand these operations. Their goals for fifth graders are limited and reasonable. Specifically, Common Core requires fifth grade students to:

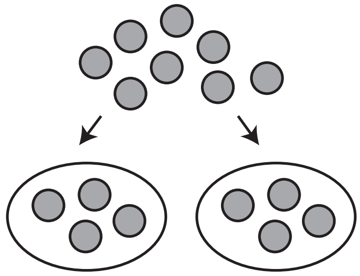
- Interpret division of a fraction by a whole number and division of a whole number by a fraction by, for instance, writing story problems to match expressions such as $6 \div \frac{1}{4}$ and $\frac{1}{2} \div 5$.
- Compute such quotients using visual models to represent and solve the problems. (Other than the expectation that students be able to write equations to represent story problems involving division of fractions, there is no call for specific numeric methods or algorithms.)
- Explain or confirm their answers by using the inverse relationship between multiplication and division (e.g., I know that $4 \div \frac{1}{3} = 12$ is correct because $12 \times \frac{1}{3} = 4$).

In order to comprehend and solve problems such as $\frac{1}{3} \div 4$ and $4 \div \frac{1}{3}$, we have to understand that there are two different interpretations of division: sharing and grouping. When we interpret division as sharing (sometimes called equal sharing, fair sharing, or partitive division), we share out a quantity equally, as shown below at left. We know how many groups we have to make; we have to find out what the size of each group is. When we interpret division as grouping (sometimes called measurement or quotative division), we know what the size of each group is; we have to find out how many groups we can make given the dividend with which we're working, as shown below at right.

$8 \div 2 = 4$

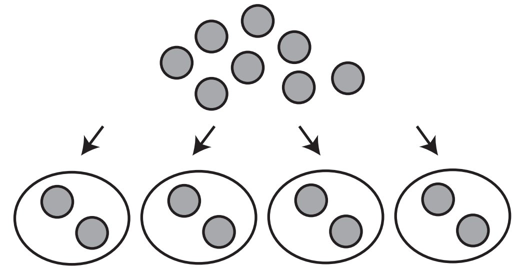
Sharing Interpretation

Here we interpret $8 \div 2$ to mean 8 divided or shared evenly, as between 2 people.



Grouping Interpretation

In this interpretation of $8 \div 2$, we determine how many groups of 2 we can make with 8.



Notice that the answer is the same in both interpretations, but it means something different in each case. In the sharing interpretation of division the result of dividing 8 by 2 tells us the size of each group;

each person getting 4. In the grouping interpretation, we already know the size of the group—2. The result of dividing 8 by 2 tells us how many groups of 2 are in 8. (There are 4.).

The importance of knowing and understanding both interpretations of division cannot be overstated because both are required to make sense of division with fractions. Consider the following: $4 \div \frac{1}{3}$. If you read this expression and try to grapple with it in any kind of sensible way, the sharing interpretation of division seems unreasonable. How do you equally share 4 things with a third of a person? On the other hand, the grouping interpretation makes better sense. How many groups of one-third can you get from 4? In other words, how many thirds are there in 4? We can reason that—there are 3 thirds in 1, so there must be 4×3 or 12 thirds in 4. We can solve the problem sensibly without resorting to inverting and multiplying. In fact, there are a couple of visual models that make it possible for fifth graders to picture and solve the problem, as shown below.

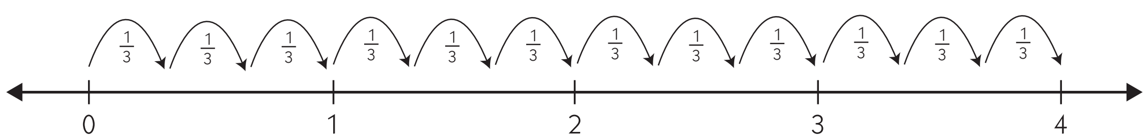
$$4 \div \frac{1}{3}$$

Grouping Interpretation of Division (Measurement or Quotative Division)

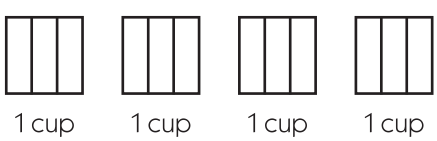
I have 4 cups of trail mix. How many $\frac{1}{3}$ cup sacks can I make with this amount of trail mix?

Basic Question: I know **what size** my groups (servings) are.
How many groups (servings) can I make?

Suggested Models: Number Line or Discrete Objects



I can make twelve one-third cup sacks with 4 cups of trail mix.



There are 3 thirds in each cup, so I can see there are 12 thirds in 4 cups. That means I can make Twelve one-third cup servings with 4 cups of trail mix.

I can also see that $4 \div \frac{1}{3} = 12$ because 12 thirds add up to 4, or $12 \times \frac{1}{3} = 4$

What about $\frac{1}{3} \div 4$? Can we use the grouping interpretation of division to help evaluate this expression? How many groups of 4 can you take out of $\frac{1}{3}$? Since that makes little sense, what about the sharing interpretation? Is it possible to divide $\frac{1}{3}$ into 4 equal shares? If you divide $\frac{1}{3}$ into 4 equal shares, each share is $\frac{1}{12}$. This may seem more difficult than figuring out how many thirds there are in 4, but a visual similar to the geoboard model students encountered in Supplement Set A9 for multiplying fractions enables fifth graders to represent and solve situations that involve dividing a fraction by a whole number, as shown next.

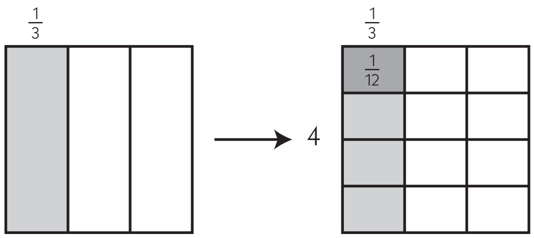
Background for the Teacher Making Sense of Division with Fractions (cont.)

$\frac{1}{3} \div 4$

Sharing Interpretation of Division (Fair Sharing or Partitive Division)
4 people are going to share $\frac{1}{3}$ a pan of brownies. What fraction of the pan will each person get?

Basic Question: I know **how many** groups (servings) are going to be formed.
What size will each group (serving) be?

Suggested Model: Geoboard, Sketches of Open Arrays (see below)



Each person gets $\frac{1}{12}$ of a pan of brownies.
I can also see that $\frac{1}{3} \div 4 = \frac{1}{12}$ because 4 one twelfths add up to $\frac{1}{3}$, or $4 \times \frac{1}{12} = \frac{1}{3}$

The pre-assessment in Activity 1 addresses the competencies Common Core expects from fifth graders in relation to dividing fractions by whole numbers and vice versa, and will give you an opportunity to see how your students do with the following skills and concepts prior to instruction:

- Solving story problems that involve dividing a fraction by a whole number
- Solving story problems that involve dividing a whole number by a fraction
- Choosing the correct operation when presented with a story problem that requires multiplying rather than dividing a whole number by a fraction
- Interpreting division of whole numbers by fractions and fractions by whole numbers

Note *If you have students who solve the problems on the assessment using an invert and multiply strategy, be aware that these children may benefit at least as much from the instruction in Activities 2–7 as those who have no way to tackle such problems yet, because the activities will give them an opportunity to make sense of an algorithm they may not really understand.*

The models and instructional strategies you use during this supplement set will lead nicely into the work students do with multiplying and dividing fractions in Grades 6 and 7. Math educators Suzanne Chapin and Art Johnson caution us, however, that some of the division situations students will encounter in sixth and seventh grade include fractions that cannot be easily be modeled using pictures or materials (e.g., $3/4 \div 2/3$*). Chapin and Johnson go on to explain that,*

> It is important to realize that not all division situations are represented by actions based on partitive division or repeated subtraction (grouping division). For example, if the area of a rectangle is 10 square centimeters and the width is $1/2$ centimeter, the length of the rectangle can be found by calculating $10 \div 1/2$. … Area is a multidimensional quantity that is the product of length and width. The "invert and multiply" algorithm, which relies on the inverse relationships between

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> multiplication and division, and between reciprocals, enables us not only to make sense of other situations but also to divide 'messy' fractions.
>
> Math Matters: Understanding the Math You Teach

So, have no doubt that there is still a place for invert and multiply, but not in fifth grade. What you do with the students this year to meet the Common Core expectations will lay solid foundations on which middle school teachers can build so their students are able to use the algorithm with good understanding.