Portland Public Schools Geocultural Baseline Essay Series

African and African-American Contributions to Mathematics

by
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Biographical Sketch of the Author

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AFRICAN AND AFRICAN-AMERICAN CONTRIBUTIONS TO MATHEMATICS

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INTRODUCTION

The purpose of this essay is to provide teachers of grades kindergarten through 12 with some examples of African and African-American contributions to mathematics. The African contribution is fundamental, extensive, but little known in our nation's schools.

Historical materials can be very useful in implementing the Professional Standards of the National Council of Teachers of Mathematics, adopted in 1991. For example, these standards state the following:

> "Tasks should foster students' sense that mathematics is a changing and evolving domain, one in which ideas grow and develop over time and to which many cultural groups have contributed. Drawing on the history of mathematics can help teachers to portray this idea: exploring alternative enumeration systems or investigating non-Euclidean geometries, for example. Fractions evolved out of the Egyptians' attempts to divide quantities — four things shared among ten people. This fact could provide the explicit basis for a teacher's approach to introducing fractions."

Educational goals that can be served by the use of historical topics in the mathematics classroom include the following:

1. To show by concrete examples that the foundations for school mathematics were created, in large part, in Africa and Asia, and to correct the false idea that mathematics is an exclusively European product.

2. To humanize the teaching of mathematics by showing the development of mathematics as a human response to human needs.

3. To help all students learn mathematics by showing that mathematics developed by going from the concrete to the abstract, much as students do in mastering mathematics.

4. To encourage students to experiment by trying methods and manipulatives from other cultures.

5. To provide role models for students and to encourage mutual respect among students of all ethnic backgrounds.

AFRICAN CONTRIBUTIONS TO MATHEMATICS

Most textbooks credit Europeans with the origin of mathematics and omit the contributions of non-Europeans. The resulting bias creates a distorted version of history. Still most teachers would like to provide their students with a more truthful picture. As a small step towards this goal, these pages will describe some of the African and African-American contributions to school mathematics.

The focus on Africa and Africans does not imply that there were no complementary or even parallel developments by other people. Some examples are the Babylonian right triangle theorem, the Chinese triangle and negative numbers, the Indo-Arabic numerals with zero place-holder, the algebra by al-Khwarizmi, the zero place-holder and base 20 numerals of the Maya, the navigational geometry of the Pacific Islanders and the mathematical knowledge incorporated into people's art.

However, Africa played a special role in the foundation of school mathematics. The widely accepted Greek tradition credited Egypt as the source of Greek mathematics, and the Egyptian priests with being the teachers of Thales and Pythagoras. The rich culture, prosperous economy, and geographical closeness for water travel made Egypt and North Africa the source and the conduit to

Europe for the ancient knowledge of Africa and Asia, including the period of high Islamic culture.

The question then arises, why was the African role in developing European civilization omitted or downplayed in our school textbooks? The great African-American scholar W.E.B. DuBois provided us with some valuable insights:

> "The rise and support of capitalism called for rationalization based upon degrading the Negroid peoples. It is especially significant that the science of Egyptology arose and flourished at the very time that the cotton kingdom reached its greatest power on the foundation of American Negro Slavery."[1]

Although chattel slavery and the slave trade that ruined Africa came to an end, in the United States sharecropping and other forms of profitable inequality followed. The European powers had divided up Africa and Asia among themselves. Racism, as an excuse for colonialism, continued to be the official policy.

This was the political atmosphere at the time that excavations in the Nile Valley were uncovering remains of the magnificent Egyptian civilization. To admit that African people had built this great civilization would have contradicted the accepted theories of racial inferiority. And so the theory of "white" Egyptians who were merely browned by the sun was invented.[2] Or, it was claimed, some non-Africans had come from the North into Egypt, a so-called "dynastic race," and it was they who created the Egyptian civilization. It must be added that these theories have been thoroughly discredited in academic circles but still influence the uninformed.

Some historians of mathematics claim that mathematics began in Greece, c. 580 B.C.E., almost 2,000 years after the beginning of written Egyptian mathematics. Everything that came before is discarded as not "true mathematics" because formal, deductive proofs had not been used. However, this essay will show that what is discarded by omitting the Egyptian and Babylonian experience is actually the origins of the main body of elementary and high school mathematics.

A thousand years passed between the end of the Hellenistic period (death of Hypatia, 415 A.D.) and the European Renaissance in mathematics. This was a rich period, characterized by Sarton as “the Muslim hegemony” in mathematics and science.[3] But Western historians, with a few honorable exceptions, have largely ignored this period of development which took place outside of Christian Europe. Europe of this period, except for Moorish Spain, remained at a relatively low cultural level.

Florian Cajori follows this version of history, which completely omits the original contributions of African and Asian mathematicians who wrote in the Arabic language. He wrote of the Islamic scholars:

> “When the love for science began to grow in the Occident, they transmitted to the Europeans the valuable treasures of antiquity. Thus a Semitic race was, during the Dark Ages, the custodian of the Aryan intellectual possessions.”[4]

Morris Kline, another widely read writer, says further that, compared with the Greeks, “The mathematics of the Egyptians and Babylonians is the scrawling of children just learning how to write.”[5]

Fortunately, in the 1960's, more balanced, comprehensive evaluations of mathematics history appeared. The 1960's were also a period of Civil Rights activism and renewed interest in African-American history as an academic subject. The mathematics writer Lancelot Hogben advanced the idea that Western civilization will increasingly have to undertake re-evaluation of a cultural debt it has hitherto amply acknowledged only to the Greco-Roman world.[6]

A few standard histories have begun to reflect views more appreciative of the African and Asian contributions to mathematics:

> “It is sometimes held that the Arabs have done little more than put Greek science into ‘cold storage’ until Europe was ready to accept it. But the account in this chapter has shown that at least in the case of mathematics, the tradition handed over to the Latin world in the

> twelfth and thirteenth centuries was richer than that with which the unlettered Arabic conquerors had come into contact in the seventh century."[8]

An especially valuable source is the work of R.J. Gillings, *Mathematics in the Time of the Pharaohs*.[9] It is the only book-length study of the ancient Egyptian mathematics to which we owe so much.

Great harm is done to African-American students and other minorities when their history is omitted or distorted. Students are left without role models. The mistaken belief that mathematics is not for them is reinforced. The damage is serious because more and more, mathematics skills have become necessary for survival. Mass-production industries have declined and the new technology requires mathematical preparation. The national interest also demands that more African-Americans, other minorities, and women enter the mathematical sciences. They will comprise almost 85% of new entrants into the workforce by the year 2,000.[10] For all of these reasons, the introduction of the African heritage of mathematics is an urgent need, not just an optional aid to mathematics education.

African Origins

The African contributions to mathematics comprise a rich field of study that includes some of the foundations for our modern mathematics. Sources of information are rich and varied — but incomplete. We do not know the whole story. Still, nothing was really lost because the development was continuous. We benefit today from all the mathematics discovered before our time. As Boyer said:

> "...continuity in the history of mathematics is the rule rather than the exception. Where a discontinuity seems to arise, we should first consider the possibility that the apparent 'hiatus' may be explained by the loss of intervening documents."[11]

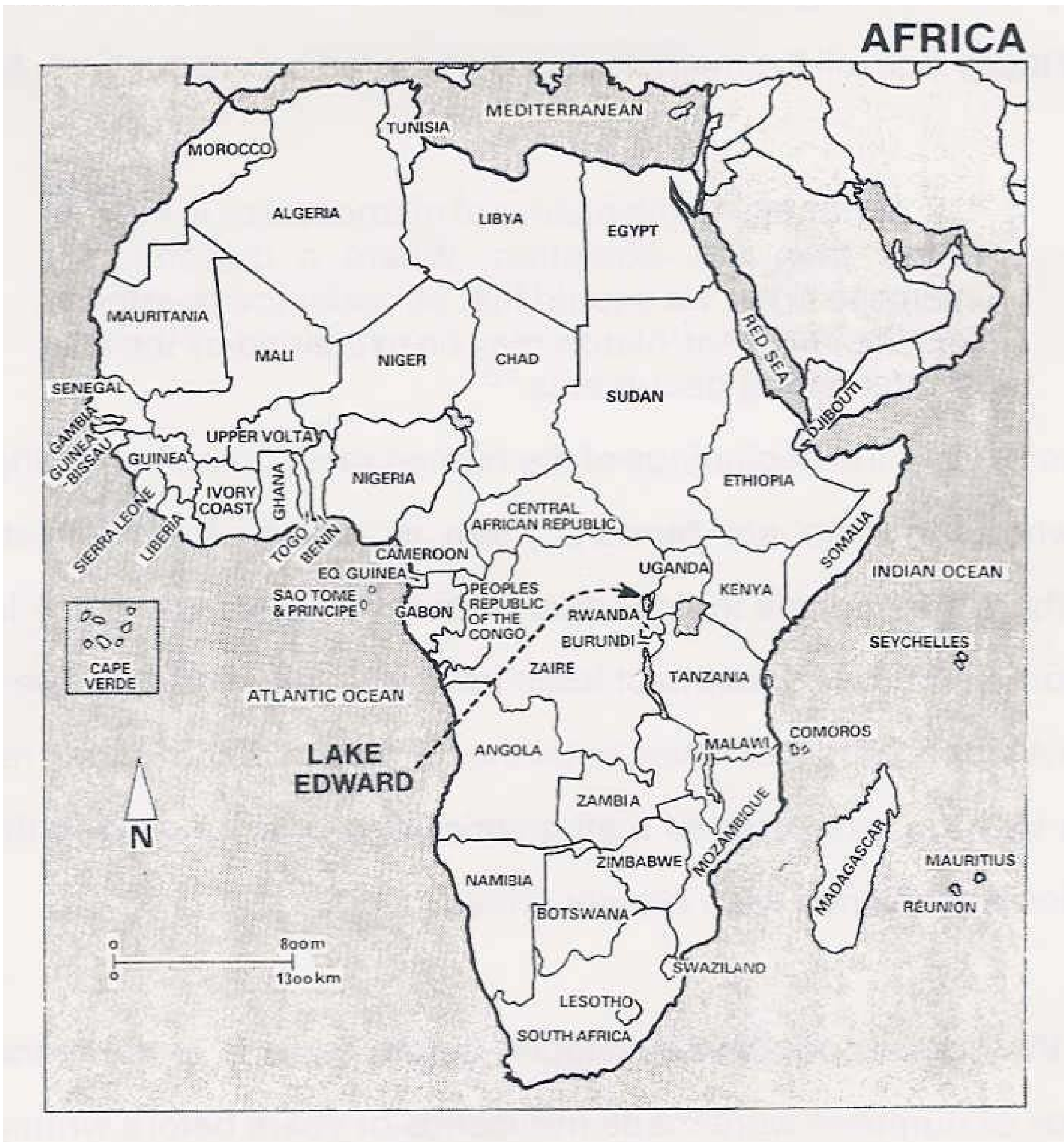
The origins of science and mathematics in Africa go back far into the prehistoric past. Most scientists believe that Africa was the birthplace of the human race.[12] In a very basic sense, chemistry began in Africa with the use of fire,[13] and mathematics began with the use of number words and logical connectives.

Certainly, the concept of number developed before records of numbers were made. And records of numbers were made thousands of years before writing developed. Tally marks were cut into bone, one mark at a time, until the count equaled the number being recorded. Not until thousands of years later, and then in ancient Egypt, were special symbols, or ciphers, used for numbers.

The earliest numerical record found to date, about 37,000 years old, comes from the Lebombo Mountains between Swaziland and South Africa.[14] It is a fossilized piece of baboon bone with 29 well-defined notches. The notches are evenly spaced and appear to represent a lunar calendar. Notched tally records have also been found in a number of countries on other continents. We do not know if tally recordkeeping spread from a common source or if different peoples thought of the same idea.

Intermediate steps were needed to advance from simple tallies to the use of abstract number symbols. The next step appears to be the grouping of tallies in some significant pattern.

About 25,000 years ago, on the shores of Lake Rutanzige (formerly called Lake Edward) between Zaire and Uganda, a pattern of tallies was carved on a bone by the Ishango people. The bone appears to be a tool handle with a bit of quartz inserted. This Ishango pattern may have influenced the later development of mathematics in Egypt. Like some entries on the Ishango bone, Egyptian arithmetic also made use of multiplication by 2.

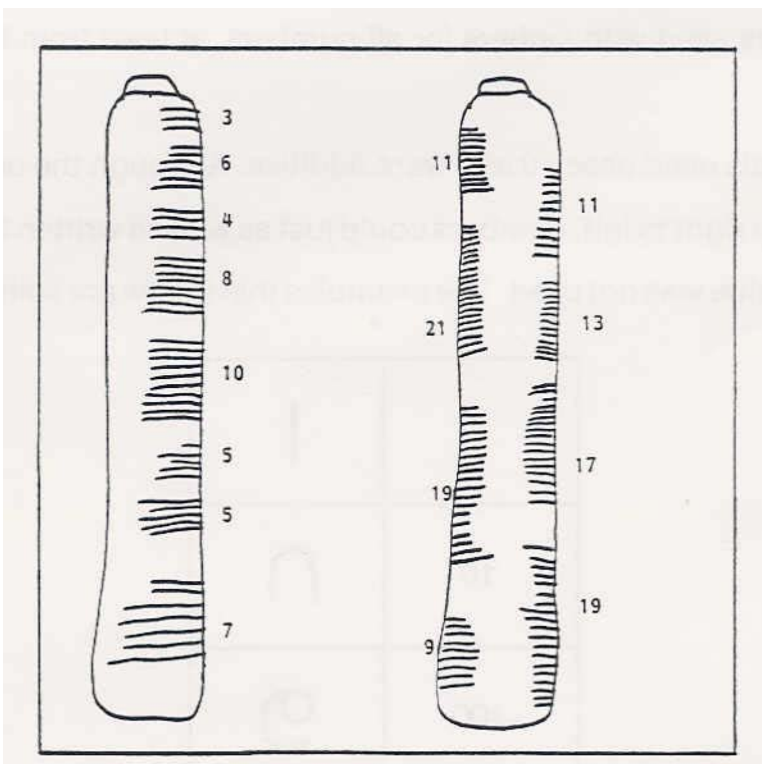


Art 1. Map of Africa

What was the mathematical significance of the Ishango bone? Tally marks carved on the bone represented numbers in the following pattern:

> "3, 6 space; 4, 8, space; 10, space; 5, 5, space; 7;
> 11, space; 13, 17, space; 19;
> 11, space; 21, space; 19, space, 9."

Some think the 3, 6, and 4, 8 suggest multiplication. Others think the sequence of prime numbers 7, 11, 13, 17, 19 is significant. Marshack believes that there is a correlation between these numbers and phases of the moon.[15]



Art 2. Two Views of the Ishango Bone

Thousands of years passed between the recording of numbers by tallies and the development of the earliest hieroglyphs from Qustul in Nubia.[16] We believe that these were years of continued development of number concepts, although little data is available. But we do know that observations of the stars and the moon were taking place, and that information was being accumulated leading to the 365-day Egyptian calendar and the 24-hour day. Although the calendar is variously dated to 4228 B.C.E. and 2773 B.C.E.,[17] Diop reminds us that many years of study of the star and moon cycles were necessary before a calendar could be formalized.[18] This process of collecting and recording astronomical data in ancient Egypt is an early example of the use of statistics for a scientific purpose.

Egyptian Numerals

A giant step for humanity was taken with the first cipherization of numerals. This step allowed the use of one symbol to represent, for example, 1,000 tally marks. Those ancient Egyptians who made the conceptual breakthrough, and used a hoop to represent 10 instead of making 10 tally marks, paved the way to the development of an efficient arithmetic. The exact date for this breakthrough is under study, but it was some time before 3000 B.C.E.

Egyptian numerals were based on 10, a forerunner of our decimal system. The Egyptian system was additive and used a different symbol for each power of 10, from 10^1, 10^2, 10^3, 10^4, 10^5, to 10^6. The Mesopotamian numerals used a place value system based on 60 and stylus-made tally marks for units and tens. The modern Indo-Arabic system used today combines principles from these two ancient systems with the important addition of a zero place-holder. The zero symbol for a place-value system was first used consistently in the Maya base-20 system. Later use of a zero in the Indian base-10 place-value numerals, and the use of ciphers instead of tallies for the digits, led to our modern numerals. They are called Indo-Arabic numerals because Islamic culture adopted the numerals and repeatedly introduced them to Europe, c. 850 to 1400. Islamic mathematicians extended this system to include decimal fractions.

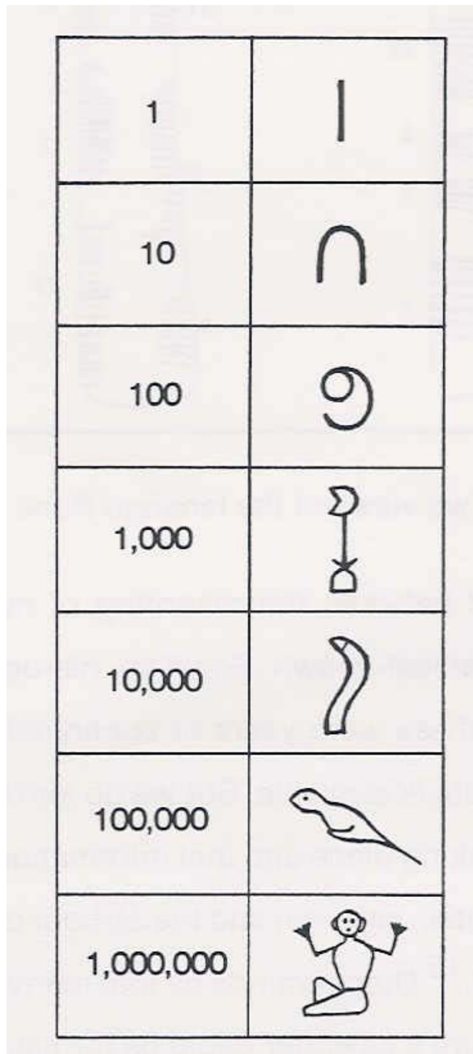
Egyptian scribes had a fast way to write, called hieratic script. Hieratic numerals were the first system in history to use complete cipherization (see Appendix B).[19]

Scribes also referred to tables to do quick calculations. For mathematics teachers, the hieroglyphic numerals provide excellent materials that can help students gain insight into the modern decimal system. The numerals shown on a page that follows can be copied, cut out, and used as manipulatives to help students develop a better understanding of the basic arithmetic operations. (Some teachers have made the Egyptian numerals from pipe cleaners to use in class as manipulatives.)

Elsewhere in Africa, other efficient number systems developed. ...However, the written record is not as full outside of Egypt. The Nubian Meroitic script has not yet been translated and little archaeological work has been done on other African civilizations. Also, we need to remember that there is more than one form of "writing," or making records. For example, the so-called gold weights of the Asante, according to Willard R. Johnson:

> "...are not best thought of as 'weights' at all....Actually they seem to have been the way the Akan people recorded their body of knowledge....They represent a miniaturization of the world and constitute a kind of encyclopedia."[20]

It has recently been demonstrated that the knotted strings of the Peruvian Inca, called quipu, are far more than a record of numbers. By the use of color coding and hierarchy of strings, as well as knotted numbers, quipus record a great deal of information.[21] In a similar manner, other peoples in Africa may well have developed record systems other than with pen or brush. This information may become available through the study of ethnomathematics which analyzes the mathematical ideas expressed in art, technology, and daily life.



Art 3. Egyptian Numerals

Most European languages show traces of counting limited to the number 2, according to Tobias Dantzig:

> "The English thrice has doubled meaning: three times and many. There is a plausible connection between the Latin tres, three, and trans, beyond; the same can be said regarding the French tres, very, and trois, three."[22]

Dantzig also refers to the San people of Southern Africa, who he says have words only for one, two, and many. But Frank Chapman reminds us that the complexity of counting systems depends on the complexity of social needs, not the intelligence of a people:

> "Abstraction, or the ability to abstract, does not imply superior intelligence. It implies that one people have had a more diverse social experience and cultural contact than another, and this is usually due to happy accident. Everyone knows what Cicero said about Anglo-Saxon slaves. But due to diverse social experiences which resulted from contact with alien cultures, these 'stupid people,' as Cicero calls them, produced such men as Bacon, Newton, and Darwin."[23]

In selecting a base for a number system, people have chosen many different bases, as in our own system which is based on 10 but has remnants of base 20 (score, as in Lincoln's "four score and seven years ago"); 12 as in the clock, dozens, and 12 inches to one foot; 60, as in minutes and seconds of time and angular-degree measure.

In Africa, there are also many number bases. Most use 5 (quinary) as their primary base and the names for two, three, and four are almost the same throughout half of Africa. "Two is usually a form of li or di. The word for three contains the syllables ta or sa, and four is generally a nasal consonant like ne. Five has a variety of forms — frequently it is the word for hand," summarizes Zaslavsky.[24]

Yoruba Number System

The Yoruba numbers of Nigeria make up an interesting, complex system that was expanded for the purpose of counting large numbers of cowries, shells used as money. Yoruba numbers use addition, multiplication, and subtraction. In our system, we use addition; for example, 131 is one hundred, plus 30, plus 1. We also use multiplication; 20 is two tens; 500 is 5 x 100, etc. But we do not use subtraction except for Roman numerals such as IX (ten less one), and time, as in five to one p.m.

The African scholar, Samuel Johnson, gives a good summary of the Yoruba numbers:

> "From one to ten different terms are used, then for 20, 30, 200, and 400. The rest are multiples and compounds. Thus, 11, 12, 13, 14 are reckoned as ten plus one, plus two, plus three and plus four; 15 to 20 are reckoned as 20 less five, less four, less three, less two, less one, and then 20. In the same way we continue 20 and one, to 20 and four, and then 30 less five (25), less four and so on to 30, and so for all figures reckoned by tens." [25]

Further, 40 is two 20's but 50 is three 20's less 10 (3 x 20 - 10). Twenties are used up to nineteen 20's (our 380), but 200 has its own name. Then multiples of 200 are used, up to ten 200's (our 2,000), but 400 has its own n ame. The next sequence uses multiples of 2,000, up to ten 2,000's or 20,000. Multiples of 20,000 provided numbers as large as were needed in the traditional Yoruba economy.

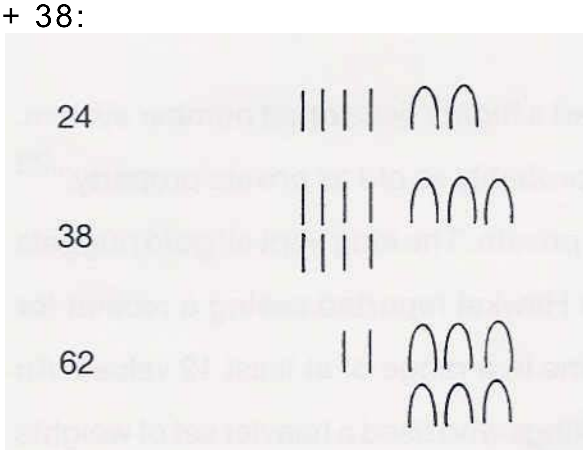
The flourishing African gold trade also required a highly developed number system. As Dantzig remarked, "Written numeration is probably as old as private property."[26] And the West African gold trade was certainly private. The king kept all gold nuggets and the people had the gold dust. In 951, Ibn Hawkal reported seeing a receipt for 42,000 gold dinars. [27] The Asante weights came in a range of at least 12 values. An indirect form of taxation was employed by the kings who used a heavier set of weights to guarantee a better than even exchange with the merchants.[28]

Cowrie shells, widely used as currency in West Africa, were counted in large quantities, requiring large numbers. When the British and other colonialists brought into Africa very large quantities of cowrie shells, purchased elsewhere at low prices, they were able to "buy" African goods at artificially low prices. As a result of the depreciation of cowries, prices jumped and units of 20,000 cowries came into use. Extensive calculations were also needed to determine the equivalency of cowries to gold. For example, 1/16 ounce of gold, or one "ackie," equaled 480 cowries about the year 1500. But 1,000 cowries exchanged for two shillings and six pence British three centuries later.

The kings of Dahomey also had an indirect tax system. The king made his purchases with cowrie strings of only 34 or 37, instead of the standard 40.[29]

MATHEMATICS IN THE NILE VALLEY

Ancient Egyptian arithmetic made use of the same principles we use today for the modern method of addition and multiplication. Addition of hieroglyphic numerals is very easy because of the additive nature of this numeral system. For example, 24 + 38:



Art 4. Egyptian Representation for Numbers 24, 38, 62

Adding units, the 4 tallies plus 8 tallies give 12 tallies. Exchange 10 tallies for a ten symbol; then put the ten symbol and the 2 remaining tallies down to the answer line. Now bring down the two ten symbols and the three ten symbols for the total of 62.[30]

A vast amount of arithmetic was used in ancient Egyptian society. It was a complex task to regulate irrigation for 600 miles of the Nile River. Just consider the arithmetic and geometry needed in the building of Khufu's Great Pyramid at Gizeh. Eves gives some impressive statistics:

> “The structure covers 13 acres and contains over 2,000,000 stone blocks, averaging 2.5 tons in weight, very carefully fitted together. These stone blocks were brought from sandstone quarries located on the other side of the Nile. Some chamber roofs are made of 54-ton granite blocks, 27 feet long and 4 feet thick, hauled from a quarry 600 miles away, and set 200 feet above the ground. It is reported that the sides of the square base involve a relative error of less than 1/14,000 and that the relative error in the right angles at the corners does not exceed 1/27,000."[31]

This summary of impressive statistics is followed by a negative conclusion: “The engineering skill implied by these impressive statistics is considerably diminished when we realize that the task was accomplished by an army of 100,000 laborers working for a period of 30 years.”

Eves' conclusion is contradictory because the employment of 100,000 laborers requires fantastic organization and planning. It should also be noted that work proceeded primarily during the flood season when the farmers couldn't work, perhaps an early form of public jobs projects for the unemployed.

The vast amount of bookkeeping needed to keep track of 100,000 people, food supplies, and building materials is staggering. Calculating wages for 100,000, at different pay scales, was not an easy task. Fortunately, Egyptian mathematics rose to the challenge.

Multiplication

Shortcuts were needed, and multiplication is a shortcut for addition, just as addition is a shortcut for counting. With the Egyptian method, this principle is easy to understand. In fact, tables were not needed in Egyptian multiplication. To illustrate: "13 hekats of grain are taken 27 times. How many in all?"[32]

Although the problem was stated in concrete terms, as hekats of grain, the numbers were thought of abstractly. The commutative law was used because it was easier to multiply by 13 than by 27.

Start with one 27 and then continue to double, which can be done simply by adding the number to itself.

1	27
2	54
4	108
8	216

At this point the scribe would stop because the next step would give 16, and only 13 of the 27's were wanted. Now the partial products are added to get 13 x 27.

\ 1	27 /	1	27
2	54	4	108
\ 4	108 /	8	216
\ 8	216 /		
	TOTAL	13	351

Students enjoy using Egyptian multiplication and it is a natural way to illustrate the distributive property:

$$13 \times 27 = (1 + 4 + 8)(27) = (1)(27) + (4)(27) + (8)(27)$$

Our current method of multiplication, brought to Europe by Arabic-speaking Africans, makes a good check for the earlier Egyptian method. This type of exercise helps many pupils realize, for the first time, the reasons behind the modern method of multiplication.

There was one exception to the doubling, or duplication, process of multiplication. Often, multiplication by 10 would be done directly. Examples of direct multiplication are shown in problems 39 and 41 of the papyrus written by the scribe Ahmose. This papyrus is known as the Rhind Mathematical Papyrus after the Scotsman who purchased it. Problem 39 asks:

"Multiply 4 so as to get 50."

	1	4
\	10	40
\	2	8
\	$^1/_2$	2
Total	$12^1/_2$ [33]	

In problem 41 by Ahmose, 64 is multiplied by 10 directly. With hieroglyphs this seems especially easy, because it would only require changing units to tens and tens to hundreds, as shown below. However, the Ahmose papyrus is written in hieratic, the fast script.

Art 5. Egyptian Representation for Numbers 1, 10, 64, 640

Inverses

Division was understood as the inverse of multiplication, as in problem 39, above. The divisor was multiplied until the quotient was obtained. Some pupils who have trouble with the modern algorithm for long division can be helped over this barrier by learning the Egyptian method.[34] In our example, we will show the subtractions, although Ahmose and other scribes never showed the details. Either Ahmose did scratch work of some type or else did the subtraction mentally.

Divide 425 by 18:

1	18
2	36
4	72
8	144
16	288
32	576

But 576 is larger than 425, our dividend. So we must go back to 16. Now from 425 subtract the partial products:

425	
-288	(16 x 18)
137	
- 72	(4 x 18)
65	
- 36	(2 x 18)
29	
- 18	(1 x 18)
11	remainder

The quotient is 23 or 16 + 4 + 2 + 1, and the remainder is 11.

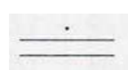
Fractions

Provisions were commonly divided among members of work crews in ancient Egypt. Divisions seldom resulted in integral quotients. The process of accurate measurement, for example in construction of the Khufu Pyramid, also requires the use of fractions. Since the Egyptian economy needed fractions, they were invented. Possibly the first fractions were parts of the cubit, the unit of length.

The Egyptians of ancient times thought of fractions as inverses of integers. Perhaps this was the reason that they always used unit fractions (numerator of 1) with the exception of 2/3. Chace, the translator of the Ahmose papyrus, observed, "They often used the fact that the reciprocal of a number, multiplying the number itself, gives 1."[35] Fractions that were not unit fractions were expressed as a sum of unit fractions. For this purpose Ahmose included a table of 2/n, where n is an odd number.

Fractions were written by putting an open mouth, the Egyptian "r" hieroglyph, over the number to show inverse. Thus, Egyptians invented the first operator symbol in mathematical history. For example, **1/8** would be:

in hieroglyphs: 

or in the hieratic script: 

Skillful use of fractions allowed the application of Egyptian mathematics to more advanced problems.[36] The scribes performed operations that were extremely complex, as we shall see below. For the addition of fractions, these ancients used an equivalent of our LCD (least common denominator).[37]

Equations

Fractions also appeared in the solution of equations. Ancient Egyptians called the unknown "aha" or heap, so that many writers call the algebra of that time "aha calculus." Equations were solved by the method of false position, a method which continued in use until the 20th century. From Ahmose comes this problem: "If 1/7 aha plus aha equals 19, what is aha?"

To find the unknown quantity, Ahmose assumed a false answer of 7. In this case 7 is a convenient choice as the least common denominator. But 7 plus 1/7 of 7 equals 8, not the required 19. To get 19 from 8, we must multiply 8 by 19/8. Then the orrection factor for the assumed false answer is 19/8. Multiply 7 by 19/8 to get the correct answer, 16 + 5/8. Of course, the Egyptians expressed the entire problem in unit fractions and wrote 5/8 as 1/2 + 1/8.

The Ahmose proof showed that he understood the necessity of proving that 16 + 5/8 **is** the solution. The proof would take 1/7 of the solution value, added to the solution, to show that the sum was 19. We would expect to see the following, using 2/7 = 1/4 + 1/28:

1	16 + 1/2 + 1/8
1/7	2 + 1/4 + 1/28 + 1/14 + 1/56

However, Ahmose wrote, "Do it thus:"

1	16 + 1/2 + 1/8
1/7	2 + 1/4 + 1/8

He must have known, whether from tables or mental arithmetic, that

1/28 + 1/14 + 1/56 = 1/8. [38]

A concept similar to the least common denominator was used to add fractions. In the above problem the common multiple would be 56. Helping numbers, written in red, were similar to our "new" numerators.

1/2 + 1/8 + 1/4 + 1/28 + 1/14 + 1/56 written in black
28 + 7 + 14 + 2 + 4 + 1 = 56 written in red

Therefore, the sum of the fractions is 1. The more complicated additions might not be the best examples for classroom use. But it surely deepens our respect for these ancient Africans to know that they correctly added 16 + 1/56 + 1/679 + 1/776 + 10 + 2/3 + 1/84 + 1/1358 + 1/4074 + 1/1164 + 8 + 1/112 + 1/1358 +

1/1552 + 2 + 1/4 + 1/28 + 1/392 + 1/4753 + 1/5432. The sum? 37! For those classrooms equipped with fraction calculators, students may enjoy checking the Egyptian accuracy in this addition of fractions.

The scribes did not explain how they completed these remarkable calculations, but they did give us some general rules. To get 2/3 of a fraction whose denominator is odd, they said, in a literal translation:

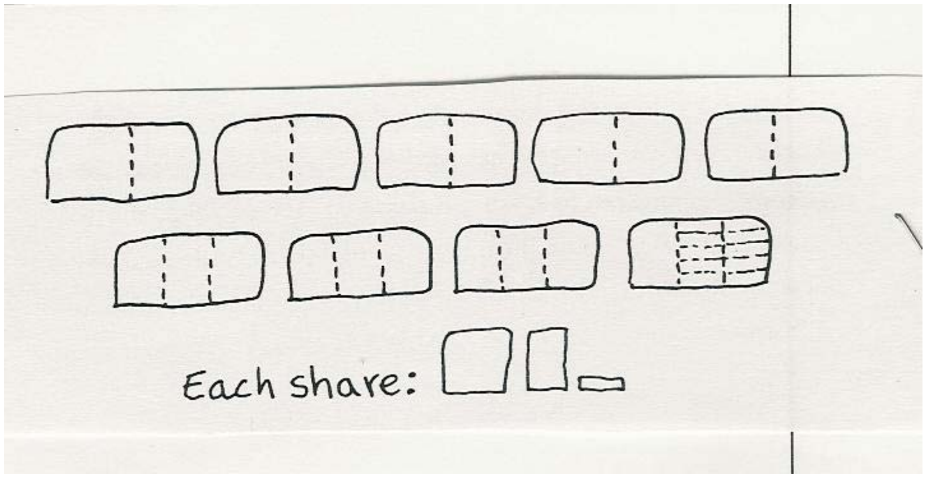
> "What is 2/3 of 1/5? Make thou times of it 2 (the denominator, B.L.), times 6 of it; 2/3 of it this is. Behold does one according to the like for every uneven fraction which may occur."[39]

Such general statements, in effect, are formulas. The specific case is this:

$$2/3 \times 1/5 = 1/10 + 1/30, \text{ and in general, } 2/3 \times 1/n = 1/2n + 1/6n$$

The complexity of these operations stumped some modern translators. Breasted judged, "Fractions, however, caused difficulty."[40] That may be true for historians, but the virtuosity of the scribes turned their fractions into a very useful tool in ancient Egypt. These same Egyptian fractions were used by scientists for thousands of years, right up to the modern period. Certainly, decimal fractions are more convenient, but unit fractions were a tremendous advance over number systems limited to integers.

Gillings had an interesting and original theory for the Egyptian motivation for the use of unit fractions.[41] As applied in the children's book *Senefer and Hatshepsut*, 9 loaves of bread can be divided equitably among 10 workers. The modern answer of 9/10 for each would leave one person unhappy with 9 separate slices. An Egyptian-type solution of 1/2 + 1/3 + 1/15 each would ensure the same sharing for each person. Students, too, can gain insight into the meaning of fractions as they result from the division of bread.[42]



Art 6. Illustration of 9 Loaves

From **Senefer and Hatshepsut**.
Permission granted by Beatrice Lumpkin.

The harmonic mean between “a” and “b”, 2ab/(a + b), provides the solution for another type of Egyptian problem. Bread was sold by pesu, which rated the amount of wheat used. If 1 hekat of wheat made 15 loaves, that bread had pesu 15
and contained less wheat than pesu 10 bread. To make pesu 10 bread, 1 hekat of wheat was used for only 10 loaves.

Suppose with 2 hekats of wheat it was required to make an equal number of loaves of pesu 10 and pesu 15. How many loaves of each should there be? The answer is not $12^1/_2$ but 12! One loaf of each would take 1/10+ 1/15 = 1/6 hekat. Therefore, one hekat would make 6 times as much, or 6 loaves of each type of bread. Two hekats would yield twelve loaves of each. This is similar to a tricky modern problem: What is the average speed of a woman who drives to a town at 20 miles an hour and returns at 30 miles an hour? The answer is 24.

Series, Arithmetic and Geometric

Many Ahmose problems involved arithmetic and geometric series arising from wage scales and division-of-bread problems connected with the complex economy and class structure. These problems also shed light on one of the greatest Egyptian achievements, the organization and administration of thousands of people into coherent work forces encompassing many trades and professions.

Ahmose problem 64 asks how to divide 10 hekats of grain among 10 men so that there is a constant difference of 1/8 between portions. Ahmose solves this problem in arithmetic series with a method equivalent to our modern formula. (Arithmetic series have a constant difference between successive terms, for example, there is a constant difference of 5 in the terms of this series: 5 + 10 + 15 + 20 + 25, + ...)

Ahmose	Steps	Formula
1. Find the average share.	$\frac{10}{10} = 1$	$\frac{s}{n}$
2. Subtract 1 from 10 to get the number of differences	9	$n-1$
3. Take half the given difference.	$\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$	$\frac{d}{2}$
4. Multiply by the number of differences.	$9 \times \frac{1}{16}$	$\frac{d}{2}\ (n-1)$
5. Add to the average share to find last, highest share, "L".	$1 + \frac{1}{2} + \frac{1}{16}$	$L = \frac{s}{n} + \frac{d}{2}(n-1)$
6. Alternate to step 5, subtract 9/16 from average share to find lowest term "a"	$1 - (\frac{1}{2} + \frac{1}{16})$	$a = \frac{s}{n} - \frac{d}{2}(n-1)$

Transposing in 6 gives the modern formula $s = \frac{n}{2}(2a + [n-1]\ d)$.

Ahmose's answer (in modern terms) was 7/16, 9/16, 11/16, 13/16, 15/16,
1 + 1/16, 1 + 3/16, 1 + 5/16, 1 + 7/16 and 1 + 9/16.[43]

Geometric series, in which the "next" term is found by multiplying the term before by a constant, were often used. In Egyptian multiplication, the multiplier was expressed by starting with 1, then using a constant multiplier of 2 to get a power series, 2^n. In Almose problem 79, the first term of the series was 7, and the constant multiplier was 7. This example resembles the Mother Goose rhyme, "As I was going to St. Ives, I met a man with seven wives... ." Ahmose's example starts with 7 houses and has been interpreted to mean that each house had 7 cats, each cat caught 7 mice, each mouse would have eaten 7 ears of grain, and each ear would have produced 7 hekats of grain.

The path from ancient Egypt to Mother Goose may have passed through Italy where Fibonacci (c. 1200), after study in Egypt, wrote:

> Seven old women went to Rome; each woman had seven mules; each mule carried seven sacks, each sack contained seven loaves; and with each loaf were seven knives; each knife was put up in seven sheaths.[44]

But Ahmose never wasted words, far from it. He merely listed two columns; we must guess the rest. Notice that the second column starts with 2,801. The addition of 1 to the sum of 7 + 49 + 343 + 2401 is not an arithmetic error, as Gillings shows, but reveals a familiarity with geometric series and their sums.[45]

Houses	7	1	2801
Cats	49	2	5602
Mice	343	4	<u>11204</u>
Spelt	2401		
Hekat	<u>16807</u>		
TOTAL	19607	TOTAL	19607

The Science of Geometry

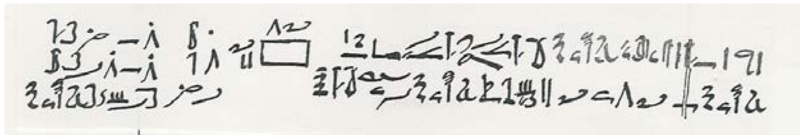
Geometry, literally the measurement of land but meaning much more, began in ancient Egypt. We must hasten to add this means written geometry, because people had studied geometric shapes for many thousands of years. The rich culture that developed in the Nile Valley was able to support the mathematicians who could afford to spend full time in the solution of problems of geometry. And, as is well known, the annual flooding of the Nile required the redrawing of farm boundaries.

Many of the Ahmose problems involve geometric formulas. Some of the most advanced formulas are found in the papyrus now in the Moscow Museum. The Ahmose Papyrus uses the correct formulas for the area of a triangle, rectangle, and trapezoid, and a good approximation for the area of a circle and the volume of a cylinder.

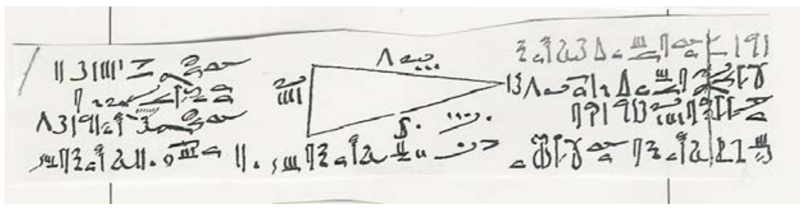
Ahmose problem 52 shows a trapezoid with bases of 6 khet and 4 khet. A third side is given as 20 khet and shown approximately perpendicular. If the third side is perpendicular, then the area by our modern formula of $A = 1/2h\ (b_1 + b_2)$ would equal 10 (6 + 4) = 100. This is the Ahmose solution:

> Add its base to its cut-off line; it makes 10. Take 1/2 of 10, that is 5, in order to get its rectangle. Multiply 20 times 5; it makes 10 (ten-setats = l00 $khet^2$, B.L.).This is its area.[46]

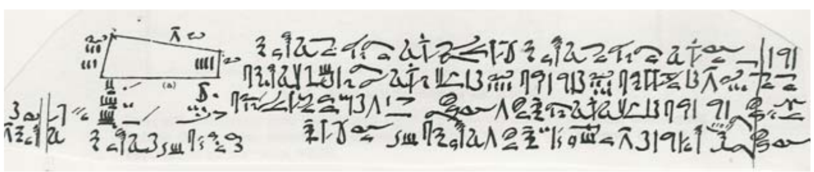
Significantly, Ahmose spoke of “getting its rectangle.” This suggests the beginning of the concept of congruence.[47]



Problem 49



Problem 51



Problem 52

Art 7. Ahmose Problems 49, 51, 52

Used with permission from the National Council of Teachers of Mathematics, from Arnold Buffum Chace, **The Rhind Mathematical Papyrus** (Reston, VA: National Council of Teachers of Mathematics, 1979).

Volume of a Cylinder

"Find the volume of a cylindrical granary of diameter 9 and height 10," asks Ahmose in problem 41. His solution contains the Egyptian value for the constant we call pi (π), a Greek letter:

> Take away 1/9 of 9. The remainder is 8. Multiply 8 times 8; it makes 64. Multiply 64 times 10; it makes 640 cubed cubits.[48]

The most interesting part of this method — volume = height times the area of the circular face — gives the Egyptian formula for the area of a circle. If D is the diameter, Ahmose takes $(D - D/9)^2$ as the area of the circle. Comparing this formula with our modern formula for the area of the circle gives the Egyptian value for π of $4(8/9)^2$ or 3.1605, an error of only 0.6%, less than 1%. Compared to the Biblical value of 3, the accuracy of the Egyptian value is very impressive.

Art 8. Area of a Circle (Ahmose Problem 48)

Egyptian Formula for Area of a Circle
From The Rhind Mathematical Papyrus, Chace, 1929

That this was not a lucky guess, but a well-established formula, is apparent from the next Ahmose example, problem 42:

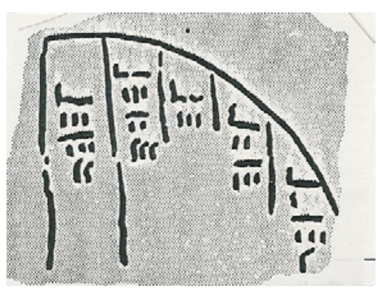
> Find the volume of a cylindrical granary of diameter 10 and height 10. In the solution, Ahmose takes away 1/9 of 10. The remainder is 8 + 2/3 + 1/6 + 1/18. He squares this quantity to get 79 + 1/108 + 1/324. Finally, he multiplies by 10.

Then there follows a lengthy change of units, as in most area problems. Clearly, the formula is the same as above.

The beginnings of trigonometry and a theory of similar triangles can be seen in several problems which find the seked of a pyramid, the ratio of the run to the rise. In modern terms, this ratio is the cotangent of the angle of inclination of the pyramid face. The values used by Ahmose are close to the seked of Khufu's pyramid of $5^1/4$ palms to a cubit, or a ratio of 5.25/7.00. Completing this triangle gives a slant side of 35/4. This is a 3-4-5 triangle.[49] Khufu's pyramid was built long before the Ahmose papyrus was written, indicating the beginning of this mathematical theory was about 1,000 years old by the year 1650 B.C.E.

Also very ancient was the practice of using square nets or grids for artwork and building plans. Then, square by square, the art detail or construction was transferred from the small sketch to the large permanent work preserving all the proportions.

Probably many construction plans were used, but only a few have come down to us. An architect's diagram of the greatest importance was found at Saqqara, dated c. 2650 B.C.E., and is much older than the Ahmose Papyrus. Just a limestone flake, about 5 x 7 x 2 inches, it gives the horizontal spacing and vertical height for points defining a curve. Nearby was a curved section of a roof, which follows the coordinates given in the diagram. Here, then, is our earliest evidence of the use of rectangular coordinates.[50]



Art 9. Egyptian Architect's Drawing, c. 2650 B.C.E.

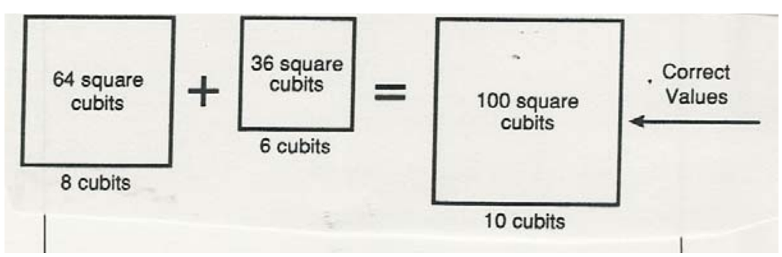
after Clarke and Englbrecht (coordinates for points on a curve).

Other examples, which looked much more like our modern square-net graph paper, were the star maps, such as found in the tomb of Rameses VII in Luxor. The position of the star is plotted against the hour of the night, with 12 rows representing the hour of the night and 8 columns locating the position of the star. [51] At an even earlier date in Egypt, there were diagonal star clocks, showing the position of the stars that were used to tell the time of night. [52]

Had we not had the luck to find that single limestone flake at Saqqara, we might not have known that the Egyptians used rectangular coordinates in their building plans. The same is true of some unique problems found in the Moscow and Berlin papyruses, known after the museums in which they were housed. Without these last two remnants, we would have been unaware of the higher level reached in these problems, which include second-degree equations and the formula for the area of a curved surface.

Second Degree Equations

Very interesting problems come from the Berlin Papyrus, named for the museum where it was housed. It was written about 1300 B.C.E., about 500 years after the originals of the Ahmose and Moscow mathematical papyri. Berlin Papyrus problem 1 asks for the size of two squares, the sum of whose areas equals a square of 100 square cubits, given that the side of the small square is 3/4 the side of the unknown square.



Art 10. `100 Cubits

If the unknown squares are of sides x and y, in modern symbols we have:

$$x = (3/4)y \text{ and } x^2 + y^2 = 100$$

Assuming the false position value of 1 for the side of the large square, then the smaller would be 3/4, and the sum of the two areas would be 1 + 9/16. Since it is the side, not the area, we are looking for, we need the square root of 1 + 9/16, which the Egyptians found correctly to be 1 + 1/4. Since the square root of the desired 100 square cubits is 10 cubits, the correction factor is 10 divided by 1 + 1/4, or 8. We get the correct values for the sides, 8 and 6 cubits, by multiplying first 1, then 3/4 by the correction factor of 8. The proof? Areas of squares side 8 and side 6 are 64 + 36 square cubits, and their sum is 100, as required.[53]

The Right-Triangle Theorem

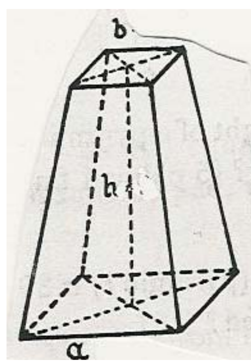
The result of Berlin Papyrus problem 1 shows that the sum of a square of side 6 and a square of side 8 is a square of side 10. This brings to mind the right-triangle theorem that carries the name of Pythagoras, a mathematician who would be born over 1000 years later. The theorem states that the square of the hypotenuse is equal to the sum of the squares of the legs. A second Berlin Papyrus problem produced two squares, one of side 12 and one of side 16 whose sum is a square of side 20. Although no theorem is stated, the values of 6, 8, 10, and 12, 16, 20 are multiples of the “Pythagorean” triple, 3, 4, 5.

Another consequence of the right-triangle relationship makes it easy to double or halve areas by changing the units of length. The basic Egyptian unit of length is the royal cubit, about 20.6 inches (recently updated to 20.7 inches).[54] For a 1-cubit square, the diagonal measures $\sqrt{2}$ cubits and is the length for another Egyptian unit, called a double remen. The remen is 1/2 x $\sqrt{2}$ cubits. By changing the unit of measurement from cubits to double remens, the area is doubled. Or changing from cubits to an equal number of remens halves the area,[55] as illustrated in Appendix A.

These specific examples indicate that the ancient Egyptians were exploring relationships of the squares of the sides of right triangles. This type of mathematical investigation probably led to formulation of the so-called Pythagorean theorem in a number of countries before Pythagoras was born. For example, the Mesopotamian Plimpton Tablet #322 contains a long list of "Pythagorean" triples. The tablet dates to 1600 B.C.E., over 1,000 years before Pythagoras. Search as one might, one can find no good reason to continue to call the theorem "Pythagorean."

Unfortunately, papyrus is a fragile material. Relatively few papyrus books survived, compared to hundreds of thousands of clay tablets from Mesopotamia.[56] In those days, the lightweight papyrus was the more convenient, but today we are lucky to have a few remnants such as the Moscow papyrus.

The Moscow papyrus, although in poor condition, reveals two of the highest achievements of the ancient Egyptian geometricians. The first was the volume of a truncated pyramid, a pyramid with the top cut off along a plane parallel to the base.



Art 11 Truncated Pyramid

Where "a" is the side of the base, "b" the upper edge, and "h" the height, the Moscow papyrus correctly gives this formula:

Volume, or $V = (1/3)h(a^2 + ab + b^2)$

Many writers have wondered how the Egyptians derived this complex formula. For example, Pottage wrote the following:

> ...If it were only a matter of explaining the discovery of a rule for the volume of a pyramid, we might be satisfied to assume that this had been deduced from the weighing of model pyramids and prisms. But the documented procedure in the Moscow Mathematical Papyrus, corresponding to our formula $V = (a^2 + ab + b^2)\, h/3$ strongly suggests more rational heuristic or justificatory procedures.[57]

Pottage,[58] Gillings,[59] Boyer,[60] and van der Waerden,[61] among others, have tried to reconstruct the ancient Egyptian method of deriving this formula. Most assume that the ancient Egyptians already knew the formula for the volume of a pyramid, $V = (1/3)a^2h$. The suggested constructions would surely make a worthwhile project for a mathematics club or science fair exhibit.

Even more spectacular is Moscow problem 10, which contains the correct formula for the area of a hemisphere. A method of confirming this formula is illustrated in *Senefer and Hatshepsut*.[62] Struve, the translator of the Moscow papyrus, said that the scribe asked for the area of a hemispheric-shaped, open basket of diameter $4^1/_2$ and found the area correctly as equal to the area of two "large circles," or twice the area of the base of the hemisphere. The formula was Area = $2(8/9)^2(D)^2$. The Peet translation, quoted by van der Waerden, changes a key word to "cylinder," a lesser problem.[63] But Gillings confirms the hemisphere interpretation, pointing out that the hemispheric baskets made in Egypt may have inspired the problem. Either way, it was a tremendous achievement, concludes Gillings.[64]

NILE VALLEY MATHEMATICS OF THE HELLENISTIC PERIOD

After the Greek conquest of Egypt, ancient Egyptian mathematics did not die but blended into the new mathematics of the Hellenistic period, built on the base of the ancient African mathematics. As Greek city-states developed, a few Greeks had traveled to Egypt to study. Thales (c. 600 B.C.E.) is credited with being the first to bring the study of geometry from Egypt to Greece. Legend has it that Thales was the first to give a deductive proof after his return from Egypt. Half a century later, Pythagoras spent over 20 years in Egypt and also visited

Mesopotamia before founding a school in Crotona, Southern Italy.[65] Democritus of Abdera (c. 400 B.C.E.) also spent time in Egypt and boasted that not even the rope stretchers of Egypt surpassed him in mathematics.

The Egyptian city of Alexandria, founded in 332 B.C.E., became the greatest center of Hellenistic mathematics. Most Egyptian mathematicians of the Hellenistic period wrote in the Greek language. But that did not make them Greek, any more than the current use of English by Japanese scientists makes them English or North American.

One of the greatest mathematicians of this era, Euclid of Alexandria, lived and worked in Africa. There is no evidence that he ever left Africa.[66] Yet he is portrayed in textbooks as a fair European Greek, not as an Egyptian. Although we have no pictures of the ancient mathematicians, we could at least visualize them honestly in costumes, complexions, and features true to the peoples and countries in which they lived. The general practice of fabricating pictures of ancient mathematicians has been called “historical forgery.” The practice of representing Egyptian mathematicians as Europeans perpetuates the myth that Europeans were the first (and only) mathematicians.

Euclid's fame rests above all on his *Elements*, containing 13 books and 465 propositions.[67] The logical arrangement of this work is so masterful that his *Elements* has dominated the teaching of geometry for 2,000 years. The deductive method of proof did more than add rigor to the largely experimental geometry of earlier Egyptians. With the deductive method, new theorems could be proved, allowing mathematics to progress beyond the immediate needs of the economy of that time. The practical side of mathematics continued, side by side with the theoretical.

The first person to measure the circumference of the earth accurately, Eratosthenes of Cyrene, Libya, was also African.[68] He measured the shadow cast by the sun in Alexandria the same day that the sun shone down a deep well in Syene, 500 miles south. The shadow showed an angle of 1/50 of a circle from zenith, directly overhead. Multiplying the 500 miles by 50 gave 25,000 compared to the modern 24,830 miles, an error of only 0.7%. Eratosthenes is also known for his “sieve” for prime numbers. The “sieve” was a procedure to systematically remove all multiples from a list of consecutive integers, leaving only the prime numbers in the list.

In trigonometry, Egyptians made important contributions, up to and including the Middle Ages. The ancients had used the concept of the seked or cotangent as a guide in building the pyramids. Menelaus of Alexandria (c.100) laid the foundations for spherical trigonometry and its application to astronomy. He was followed by another great Egyptian, a half century later, Ptolemy of Alexandria, author of the *Almagest*. Known as an astronomer, his work in trigonometry alone would have ensured his fame. To aid him in the extensive calculations needed for his astronomical tables, Ptolemy developed formulas for the sines and cosines of the sums and differences of two angles and half angles. His tables remained in use for 1,000 years. Ptolemy also improved on the excellent ancient Egyptian approximation of π with his value of 377/120 377/120 $\cong$ 3.14167.[69]

In this same period, Heron of Alexandria invented 100 machines and wrote many mathematical works. Heron and Diophantus, both of Alexandria, are often acknowledged as Egyptians who worked in the Egyptian tradition. This tradition is wrongly described as limited to applied mathematics. Of course, as Alexandrians, they were Egyptians, but their work had theoretical as well as practical interest. Diophantus' methods had been criticized as not sufficiently general in finding rational solutions of indeterminate equations. But the historian Bashmakova showed that “...the essence of Diophantus' method was concealed by the fact that he solved only separate problems with given numerical data, but without formulating his methods in general. In reality, these numbers played the role of parameters.”[70] Diophantus' work inspired Fermat's “last theorem” — that there are no positive, integral values (given n greater than 2) for x, y, and z such that $x^n + y^n = z^n$.

Longer than any other city, Alexandria endured as a scientific center. Its last days of the Hellenistic period were highlighted by the short, brilliant career of Hypatia, a woman algebraist who held the chair of the department of philosophy at the

University of Alexandria. In 415, a fanatical religious mob brutally murdered Hypatia, literally tearing her apart. Some textbooks depict Hypatia as a white European, although she was born in Egypt, the daughter of Theon, also an Egyptian. Her prominence as a department chair was certainly in the Egyptian tradition of greater rights for women as compared to the near slave status of Greek women.

The Alexandria of Euclid, Ptolemy, Heron, Diophantus, Hypatia, Pappus, Menelaus, Theon, and many more was nourished by the productive agriculture of the Nile Valley as well as the commerce between the Nile and the Mediterranean. Alexandrian mathematics was a blend of the more ancient Egyptian traditions and the newer Greek mathematics.

After Hypatia was murdered in 415 and Proclus died in 485, any Alexandrian mathematics that may have continued did not come down to us. But by the year 750, Islam began to revive centers of learning in Northern Africa. Euclid's *Elements*, Ptolemy's *Almagest,* and other volumes appeared in Arabic. In the Nile Valley, a new center of learning arose in the city of Cairo. But in Europe, little if anything was left of the mathematical schools. For lack of anyone to read and understand the great classics, even the books were lost.

AFRICAN ISLAMIC MATHEMATICS

In the 9th century, arithmetic and algebra textbooks written in Arabic revolutionized mathematics. Eventually, through trade between North Africa and Europe, the books reached Europe and were translated from Arabic to Latin. Some cities passed laws forbidding use of the new Indo-Arabic numerals, claiming that they were too easy to alter for the purpose of fraud. Still the new numberals (basically the same numerals used today) began to displace the clumsier Roman numerals.

Our word for algebra was taken from the name of the algebra text. *Al-jabr wa'l muqabalah*. From the Central Asian author's name, al-Khwarizmi, came our word for algorithm, a mathematical procedure. Al-Khwarizmi stated that his purpose in writing his book was to serve the practical needs of the people concerning matters of inheritance, legacies, partition, lawsuits, and commerce.[71]Half a century after al-Khwarizmi, a commentary written about al-Khwarizmi's book further advanced the concepts and techniques of algebra. It was written by Abu Kamil , a mathematiciian known as the "Egyptian calculator," (850-930). His work influenced mathematicians for centuries and was copied wholesale by Leonardo of Pisa (Fibonacci) 300 years later. Fibonacci is famous, but few today know the name of the great African mathematician, Abu Kamil.

Abu Kamil's work included use of several variables — al-Khwarizmi was restricted to one — a study of equal roots of quadratic equations, and especially the use of irrational numbers as terms of proportions, roots, and coefficients of equations. For example, to solve the following system:

$$x + y + z = 10$$

$$xz = y^2$$

$$x^2 + y^2 = z^2$$

Abu Kamil used the false position value of x = 1. This led to

$$x + y + z = 1 + 1/2 + \sqrt{1 + 1/4} + \sqrt{1/2 + \sqrt{1 + 1/4}}$$

Since the right side should have been 10, Abu Kamil set up a proportion to find the correction factor needed. And all of this was done with words, without the advantage of modern symbols, square root signs, etc.[72] Diophantus, an earlier Egyptian, had introduced symbols for squares, cubes, etc. But his lead was not followed until much later. Abu Kamil called the unknown the "thing." For x^2 he used the word "square." Then x^4 he called "square square," x^6 "cube cube," and x^8 he called "square square square square."[73] This terminology uses the addition of exponents in multiplication. Indeterminate equations were the subject of another work by Abu Kamil, his *Book of Rare Things in Arithmetic*.[74]

Abu Kamil presented 20 geometric problems in *The Decagon and Pentagon*, a work copied extensively by Fibonacci, 300 years later.[75] In one problem Abu Kamil asks for the side of a regular decagon whose area is 100 — a specific value. Yet his method is general and his approach is almost entirely algebraic. Step by step, Abu Kamil finds the unknown side x from $x^4 = 1600 - \sqrt{2{,}048{,}000}$ (approximately 3.6).[7]

Just a few years after Abu Kamil, Egypt came under the rule of the Fatimids (969-1171). The power of Egypt extended from North Africa to Syria and Western Arabia. There was a general surge of rapid development, economic as well as cultural. Windmills were built in Egypt and other new devices came into use. Yushkevitch, a Soviet historian of mathematics, wrote that all of the sciences flourished; chemistry, medicine, pharmacology, zoology, botany, and mineralogy knew an extraordinary development.[77] Cairo, founded in 969, became the capital, and the site of a science academy, the Dar el Hikma, or House of Wisdom.

A very well-equipped observatory was built on the Mukattam Heights. There Ibn Yunus, considered by astronomers of his time as the greatest astronomer, worked on his famous Hakimi Tables, which included observations of eclipses and conjunctions of planets.

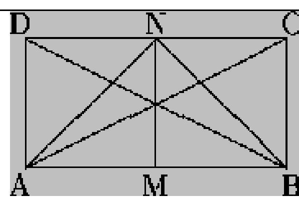
Ibn Yunus improved on work that Ptolemy had done about 900 years earlier. In trigonometry, Ibn Yunus was also credited with the prosthapherical formula that was used to simplify lengthy calculations. In fact, that formula may have been developed by another Islamic mathematician, but Ibn Yunus left more than enough work to ensure his permanent place in history. This formula states the following:

$$\cos A \cos B = 1/2\,(\cos[A + B] + \cos[A - B])$$ [78]

For those who doubt that something called prosthapheresis could really simplify anything, this formula converts multiplication to addition, a simpler operation.[79] Indeed the same method was used by the famous Danish astronomer Tycho Brahe (1546-1601) and became known to Napier in Scotland. Then Napier used this principle to invent logarithms.[80] Ibn Yunus was able to calculate the sine of 1°

accurately to four decimal places. So accurate were Yunus' calculations that he succeeded in developing a table of sines for angles differing by just one second.[81] This was the sine function that medieval Islam had received from Indian astronomers. In Baghdad, Abu al-Wafa had calculated the six modern trigonometric functions.

At the same Cairo Science Academy where Ibn Yunus worked, the Iraqi-born Ibn al-Haytham (died c. 1039) did the crowning work of his career. Haytham was famous for his work in optics.[82] Less well-known, but of the greatest importance, were Haytham's contributions to mathematics. He developed a formula for the sum of a series of 4th powers which enabled him to evaluate the equivalent of the integral $\int t^4 dt$. His original work in geometry was also developed by other famous Islamic mathematicians, especially Umar al-Khayyami and Nasir al-Din al-Tusi. They laid the basis for modern non-Euclidean geometry, which describes the curvature of space. It is a fact that Saccheri, the European pioneer in this field in the 18th century, used the work of Islamic mathematicians. The “Saccheri quadrilateral” closely resembled Ibn al-Haytham's quadrilateral of 700 years earlier.[83] (Haytham was called Alhazen in Latin.)



AB is the base and CD the summit. Angles A and C are right, and AD=BC. The Summit Angles of a Saccheri Quadrilateral, so defined, are Equal and Acute. *The line joining the midpoints of the base and the summit--called the **altitude** --is perpendicular to both.*

Art 12. Saccheri Quadrilateral

Important discoveries were still made by Islamic mathematicians as late as the 15th century. Included among these Islamic scholars were Jews and Christians as well as Muslims from many different countries. In fact, as early as 950, al-Uqlidisi of Damascus solved problems with the use of decimals.[84] Al Qasadi of Granada, Spain, who died in Africa in 1486, did remarkable work in bringing symbols into algebra, before this development began in Europe. He used the first letters of the Arabic words to show unknown, square, or cube, and a symbolic equal sign. The symbolization was so advanced that Yushkevitch believes there were other Islamic mathematicians before al-Qasadi, who began this development.[85] Yet this outstanding work is still little known outside of the Arabic-speaking world.

The contributions described above have led some historians to revise the standard Eurocentric evaluation — that the Islamic mathematicians added nothing new, and served only to preserve the classics of the Greek mathematicians. This type of bias is expressed by Morris Kline, that mathematics “finally secured a firm grip on life in the highly congenial soil of Greece and waxed strong for a brief period....With the decline of Greek civilization, the plant remained dormant for a thousand years...when the plant transported to Europe proper and once more imbedded in fertile soil.”[86]

J.F. Scott, admits that Islamic mathematicians “did more than preserve; they made some significant contributions of their own.[87] But two pages later, the same writer, in the same book, declares, “The debt which the West owes to the Arabs for their part in preserving and transmitting Greek science is very great. It must not be forgotten, however, that preservation is one thing; creation is something different. Mathematics for its development requires the creative faculty, and there is little evidence of this in the many centuries which separate the decline of Alexandrian science and its revival in the West.”[88]

Such inaccurate views are refuted by the same Europeans who borrowed so heavily from the culture of Africa and Asia in the Middle Ages. For example, Leonardo Fibonacci of Pisa wrote:

> “All that was studied in Egypt, in Syria, in Greece, in Sicily, and in Provence...I investigated very carefully...I wanted to write a work of 15 chapters, with nothing capital left without a demonstration and this I did so that the science might be easily understood, and the Latin people should no longer be deprived of it.”[89]

PROGRESS HALTED

When Fibonacci wrote these words, Egypt, compared to Italy, was more advanced in science and culture. But unfavorable changes during the rule of the Turkic Mamelukes and the Ottomans slowed down Egyptian development. With the strengthening of feudal structures, the economy of the prosperous Islalmic states retrogressed and scientific output declined. While the growth of capitalism and industry was being stifled in Egypt, the merchant capitalists in the city states of Europe were gaining power. Western Europe was also spared the damaging invasions suffered by North Africa and Western and Central Asia.

Still, there was a rough parity between Western Europe and many African states in the 15th century. This equality was destroyed, and the economies of Africa were devastated by the slavery and colonialism that followed. Mathematical and scientific output slowed to a halt with the disruption of the Nile Valley economy. Even the memory of these achievements was almost destroyed by a flood of misinformation and racism, let loose to justify slavery and imperialism. Still, some African universities continued to function south of the Sahara for some hundreds of years. These were protected from European disruption because of their inland locations.

Timbuktu in ancient Mali, later part of the Songhai Empire, was an international center of learning in the 14th to 16th centuries. Leo Africanus, the North African historian, reported that the book trade of Timbuktu brought in more money than any other commodity.[90] This tradition was carried over to Katsina, a Hausa state in Northern Nigeria. There Muhammad Ibn Muhammad wrote works on number theory which unfortunately were lost. But we do have a copy of his work on magic squares, c. 1732. When an English traveler, Hugh Clapperton, visited the Sultan of Northern Nigeria in 1826, he found him studying Euclid's *Elements*.[91]

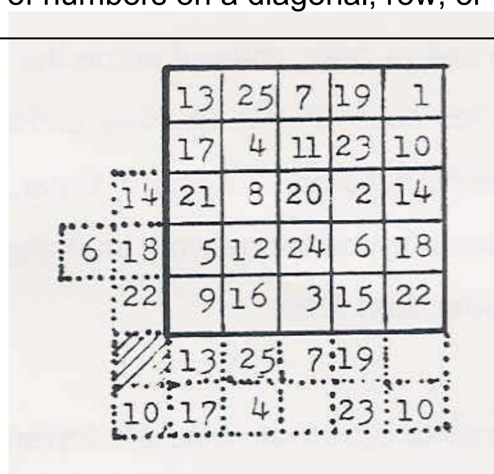
AFRICAN MATHEMATICAL GAMES

Muhammad Ibn Muhammad devised several methods for constructing magic squares of odd order, 3 x 3, 5 x 5, etc. One of his ingenious methods used chess knight moves of two boxes down and one to the left. Muhammad constructed this 5 x 5 square with this method, as illustrated in Claudia Zaslavsky's excellent discussion of African mathematical pastimes in *Africa Counts*.[92]

The additional boxes (dashed lines) represent an imaginary “wrapping” of the figure around a cylinder, so that top and bottom edges meet, also right and left sides. For the 5 x 5 square, the next move after the numbers 5, 10, 15, 20 is two boxes to the left.

Starting with 1, a knight’s move of 2 down and 1 to the left gives the location of 2. Then 2 boxes down and 1 to the left locates 3. The extensions (dashed lines) are needed to get to 4 in the dashed box which “wraps around” to be placed in the second row.

In this square, the sum of numbers on a diagonal, row, or column is always 65!



Art 13. Numbers Grid

A similar method was introduced to Europe from Spain, which remained a conduit for African knowledge long after Moorish rule ended.

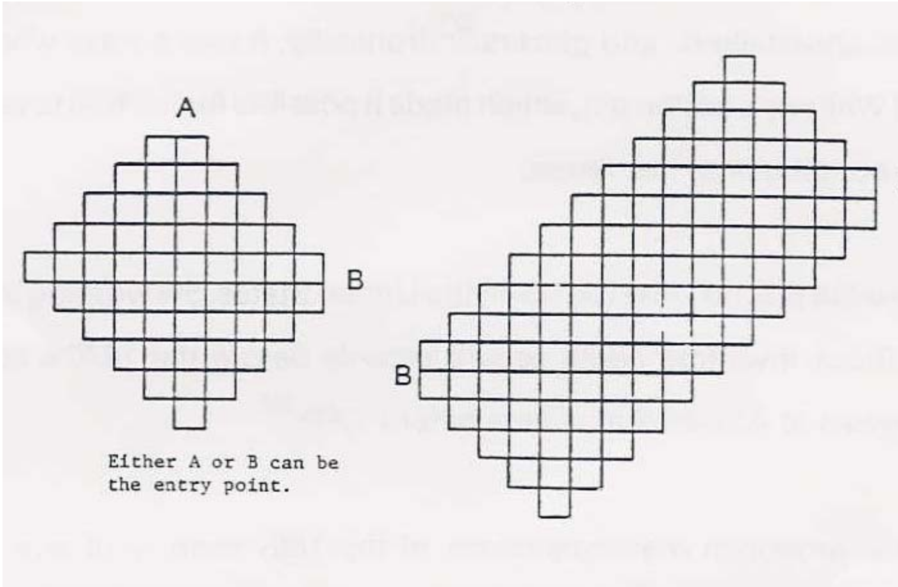
Another mathematical game played throughout Africa, from very ancient times to the present, was witnessed by this author in Maputo, Mozambique, Southeast Africa. Described by the anthropologist Henri Jounod as an elegant game, the Mozambican version is excellent for classroom use because it allows teamwork.[93] Called “N’tchuba,” and played with four rows of holes or cups dug in the earth, it can as well be played on a table with one to five persons on a side. The game teaches quick mental arithmetic, addition, subtraction, and also the concept of "clock arithmetic," or cyclical numbers. Detailed rules and a sample game are outlined in Appendices F and G.

We have discussed algebraic skills involving numbers, and working with fixed geometric figures. There are other mathematical skills involving reflection, translation, and rotation, transformations which are well-developed in African art and games.

The Bushoong children of Central Africa (Zaire) draw complex networks which can be traced with a continuous line or path, without lifting the pen or doubling over a line. Rules for solving puzzles of the type enjoyed by the Bushoong children for many centuries were developed by the great Swiss-born mathematician Leonhard Euler in 1735. Euler established a new branch of topology called network analysis.

Briefly stated, network analysis tells us that paths are traversable if they have even vertices only, that is, an even number of lines coming in and going out at each vertex point. They are also traversable if they have exactly two odd vertices, in which case one is an entry point and the other, the exit point. Otherwise, networks are not traversable. That proved to be the case for the famous puzzle of traversing the seven bridges in the city of Koenigsberg. Euler showed that the bridges were not traversable.[94]

Can you traverse these figures which Bushoong children draw in a continuous line, without lifting their finger or doubling back on a line? They are two of many adapted in Zaslavsky's book[95] from the work of Emil Torday.



Art 14. A/B Entry Points

AFRICAN-AMERICAN MATHEMATICIANS

The disruption of African culture by the European slave trade and colonialism has been described earlier. The children of Africa were scattered far and wide (those who survived the cruel passage), many in the United States. What has been their mathematical experience in the United States?

The inventive spirit was never totally annihilated during the days of American slavery. American slavery was especially vicious because it was tied to a capitalist market with an insatiable appetite for profit. The slaves were exposed to a diabolic and consistent application of mental and physical torture.[96] Still, many slaves retained the craft skills they brought with them from Africa or learned in the United States. Use of these craft skills opened the door to quantitative thinking and invention.

Skills of African-Americans, slave and free, included the trades of coopers, tailors, bankers, tanners, goldsmiths, carpenters, sailors, sailmakers, naval carpenters, blacksmiths, weavers, millers, masons, candlemakers, tobacconists, caulkers, cabinetmakers, shoemakers, and glazers.[97] Ironically, it was a slave who provided the idea for Eli Whitney's cotton gin, which made it possible for “cotton to be king” but worsened the conditions of the slaves.

By 1860, there were 500,000 free African-Americans in the United States, 5% working in industry. Although no African-American inventors could secure patents before the 1870's, by 1900 they were granted at least 350 patents.[98] Mathematical skills were needed to draw the diagrams for successful patent applications.

Benjamin Banneker and Thomas Fuller

One of the few American mathematicians of the 18th century was Benjamin Banneker (1731-1803), the son of a freed African. At a country school he received a few months of basic education but was primarily self-taught as a mathematician, clockmaker, surveyor, and astronomer. Thomas Jefferson heard of Banneker's skills and called him to Washington to be the astronomer on the team of surveyors that would draw up plans for the nation's new capital.

Jefferson, then Secretary of State, sent Banneker's *Almanac* to the Academy of Science in Paris, as an example of the scientific potential of African-Americans. Banneker's letter to Andrew Ellicott reveals Banneker's motivation for his studies: “...the Calculations were made more for the sake of gratifying the curiosity of the public, than for any view of profit, as I suppose it to be the first attempt of the kind that ever was made in America by a person of my complexion...I find by my calculation there will be four Eclipses for the ensuing year.”[99]

Although Banneker spent most of his life doing hard work on his farm, he never gave up his interest in mathematics and astronomy. At the age of 22, Banneker showed his potential genius by making a clock with only a borrowed pocket watch as a model. The clock kept perfect time throughout Banneker’s life and faithfully struck the hour. Benjamin Banneker became famous and people came from miles around to see his clock.

When Banneker was 57 years old, a neighbor gave up his engineering career to concentrate on business. The neighbor loaned his books and instruments to Banneker. Without a moment's instruction, Banneker started his study of algebra, spherical trigonometry, and the astronomy of that time. He also learned to do the 63 calculations and the 10 geometric projections needed to predict eclipses. His skill progressed so rapidly that he decided to compile his own almanac, the work that brought him to the attention of Thomas Jefferson and eventual fame.[100]

Another African-American, Thomas Fuller, lived about the same time just a few miles from Banneker in the neighboring state of Virginia. Unfortunately they did not know each other because Fuller was a slave and Banneker had no money for traveling. Fuller could have been a big help in the calculations for Banneker's almanac.

Fuller was also a mathematical genius and could mentally multiply two 9-digit numbers. He was very fast and always accurate. Stolen from Africa at the age of 14, Fuller was enslaved for the rest of his life and had no chance to learn to read or write English. Entirely self-taught, he was challenged to find the number of seconds a man has lived who is 70 years, 17 days, and 12 hours old. The challenger was a white scholar sent to test Fuller. The scholar represented the abolitionists, people who wanted to abolish slavery. Abolitionists hoped to use Fuller's genius as an example to refute the false theories of racial inferiority.

The challenger spent ten minutes calculating with pencil and paper. His answer was 2,209,032,000. "Too small," said Fuller. "You left out the leap years!" Fuller gave the correct answer, 2,210,500,800, in less than two minutes![101]

Despite the 13th Amendment, which legally ended slavery, legal racial discrimination continued in many states, up through the 1960's. This oppressive atmosphere not only discouraged the mass of African-Americans from entering mathematical studies, but, most unfortunately, penetrated the mathematical organizations themselves, such as the MAA (Mathematical Association of America) and the AMS (American Mathematical Society). Lee Lorch, then at Fisk University, led the fight to eliminate segregated seating and other discriminatory practices from the professional mathematical societies.[102]

A significant number of African-American mathematicians overcame all the hurdles and achieved the Ph.D. in mathematics. The first was Elbert Francis Cox (1896-1962), who received his degree from Cornell University in 1925, when only 28 doctorates in mathematics were awarded in the entire United States. Cox was head of the mathematics department at Howard University for 32 years. Next was J. Ernest Wilkins, who received his Ph.D. in 1942 from the University of Chicago at age 19. He worked on the Manhattan Project and later became a professor of applied mathematics and physics at Howard University. About the same time David Blackwell received a Ph.D. from the University of Illinois and became the first African-American elected to the National Academy of Science.

The need to change teachers' attitudes was underscored by Walter R. Talbot, the fourth African-American Doctor of Philosophy in Mathematics:

> "Nowadays our promising youth are even more menacingly threatened by exposure to teachers who have not only been vigorously and successfully indoctrinated relative to the difficulty of mathematics, but also have been convinced to their viscera that Blacks, however successful in sports, music, politics, law, medicine, and so on and so on, are abysmally and irrevocably hopeless as far as mathematics is concerned."[103]

The widow of William Clayton, one of the first African-American mathematicians at the Ph.D. level, explained how discrimination blasted his career:

> "Writing about Bill and his presentation at the Math Society, I thought about the days Bill used to tell me how owing to the Black-white mess, he had to stay at a private home when the others were at the hotel where the Association met. Over the years when the color line became less, he never would attend any more meetings . . . I guess the hurt went too deeply with him."[104]

A sign of the changing times is an article in the same Mathematics Association journal, "Black Women in Mathematics in the United States,"[105] written by Patricia Kenschaft. In 1980 there were over 26 African-American women in the United

States who held the Ph.D. in mathematics. The first were Marjorie Lee Brown, University of Michigan, and Evelyn Boyd Granville, from Yale, both in 1949. As an illustration of the very great difficulties overcome by African-American women mathematicians, Kenschaft reports that Granville applied for a faculty position at an institution left unnamed. When the hiring committee discovered that Granville was African-American, they merely laughed at her application and never considered her for the job, even though she had a Ph.D. degree from Yale.

Granville taught at Fisk University from 1950 to 1952. At least two of her students, Vivienne Malone Mayes and Etta Zuber Falconer, also went on to earn doctorates in mathematics. Granville also worked in space technology for IBM on Projects Vanguard and Mercury. She did research on space trajectory and orbit computations, numerical analysis, and digital computer techniques. A third Fisk caring student, Gloria Conyers Hewitt, completed the doctorate in mathematics. Mayes, Falconer, and Hewitt paid special tribute to their professor, Lee Lorch, as an example of the contribution teachers can make to help students excel in fields formerly closed to them by race or sex prejudice.

Although space does not permit mention of each of the distinguished African-American scholars who are professional mathematicians, the life story of Eleanor Green Dawley Jones is especially instructive since she had the responsibility for small children at the same time she was doing her graduate work. As told by Kenschaft:

Eleanor Green Dawley Jones

> "...was educated in completely segregated schools and the only whites she knew as a child were the priests and nuns of her parish. When she graduated from high school at the age of 15, she won a scholarship to Howard University. There she studied under Elbert Cox, the first Black American to receive a Ph.D. in mathematics, as well as several other Black men with doctorates. She received her B.S. in 1949 and her M.S. in 1950, both in mathematics. For a while

> after her graduation she taught in high school. In 1955 she became an instructor at Hampton Institute.
>
> "The Black men she had known with doctorates served as role models, and it occurred to her that she should work for a doctoral degree in mathematics at Syracuse University. At that time no Black people were allowed to pursue doctoral studies in any academic discipline in Virginia, but the state would pay tuition and travel costs of Black citizens who went out of the state for graduate study.
>
> "Jones by then had two small sons, whom she took with her to Syracuse. There she earned her own living and that of her children while obtaining her doctorate which she received in 1966. Her thesis advisor was James Reid and her thesis topic was 'Abelian Groups and Their Endomorphism Rings and the Quasi-Endomorphism of Torsion Free Abelian Groups."[106]

In 1984, NASA awarded its greatest prize, the Space Act Award, to David R. Hedgley, an African-American mathematician. His major breakthrough in computer graphics was a code which allows a wide variety of computers to show three-dimensional shapes. As reported in the *Chicago Defender,* "The Hedgley solution is general, very fast and applicable even in complex three-dimensional scenes."[107]

In 1990, the National Science Foundation (NSF) published an inspiring book, *Models of Excellence*. This book includes the stories of over 100 minority scientists and mathematicians who are conducting NSF-supported research at the post-doctoral level. Included in this sample are 11 African American mathematicians, and 22 African American scientists in the physical sciences. Although the number of minority mathematicians is increasing, many, many more are needed to achieve the goal of full equality by the year 2000. There are, however, thousands of African-Africans employed in mathematics-based professions.

SUMMARY

In this short space, many important topics could not be included, such as the use of geometry in African architecture. For example, the Great Zimbabwe stone structures of the Monomotapa empire and the Ethiopian churches carved into solid rock made use of geometric design and careful calculation. Future archaeological work will reveal more information on African mathematical achievements.

Still, the materials in this essay have supplied sufficient information to show that the main body of elementary and high school mathematics was developed in Africa and Asia. The later European development of mathematics rests on an African and Asian base.

Today, an increasing number of African-Americans have become creative mathematicians, but many more are needed. The knowledge that mathematics is art of the African heritage, and that the African heritage is a very important part of the entire human heritage, should inspire many more students to enter the mathematics professions.

APPENDIXES

Appendix A

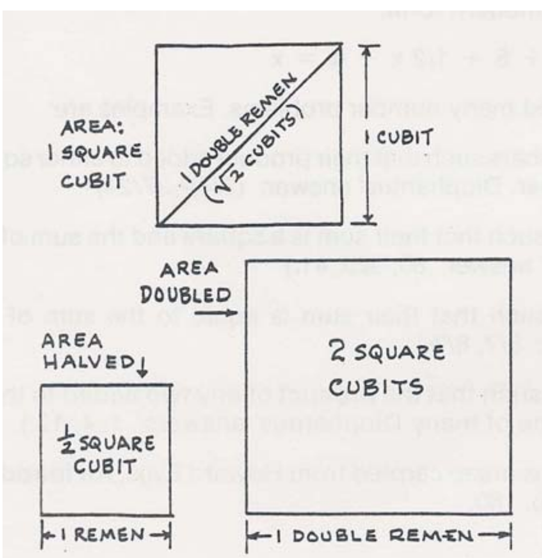
A: Egyptian Units of Measurement

Scales and weights go back to prehistoric times in Egypt, and a system of weights was well-established in Egypt, as in other African countries. The basic unit of weight, the deben, was a decimal measure with 10 kite = 1 deben. The deben was about 91 grams, making the kite about 9.1 grams.

Some of our basic units of time come from Egypt, such as the 365-day year and the 24-hour day. The Egyptians were also the first to make the hours of the day equal in duration.[108] Star clocks, water clocks, and sun dials were also invented in Egypt.

The cubit, length of a royal forearm, became the standard unit of length throughout Africa and near-Asia river civilizations, and made its way to Europe. The smaller units were: 4 fingers = 1 palm; 7 palms = 1 cubit (royal).

Other "cubit" measures, the remen-cubit and the double-remen cubit, were in wide use, and relationships shown in the following diagram could serve to double or halve areas.



The angle of inclination of a pyramid was measured indirectly, by its "seked" (run to rise) or cotangent. Ahmose gives this value as 5 palms 1 finger per cubit.

Art 15. Egyptian Units of Measure

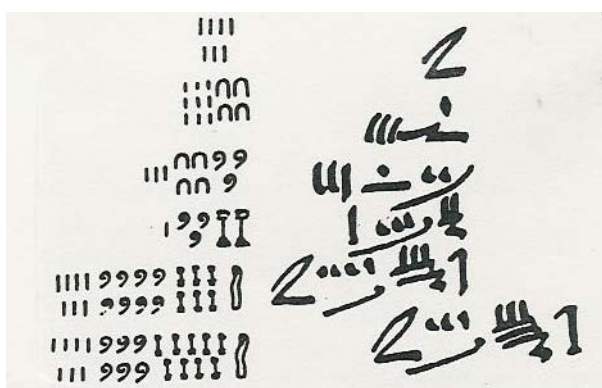
From ***Senefer and Hatshepsut***, used with permission of Beatrice Lumpkin.

Appendix B

B: Cipherization and a Symbol for Zero

The cipherization of numerals and the development of the zero concept are little known African achievements. In hieroglyphic numerals, values were expressed by grouping and addition of repeated ciphers. Numerals did not have positional value. Most scribes, however, used the cursive, hieratic numerals. For example, 19,607 written in hieroglyphs would require 23 symbols, but in hieratic would need only 4 symbols. That's because the hieratic numerals used a different principal. As Boyer explained, a unique symbol was used "for each of the first nine integral multiples of integral powers of ten." He called the hieratic numeral system, "decimal cash-register cipherization," referring to old-style cash registers which sent up a flag for each decimal place. Boyer added that, "The introduction by the Egyptians of the idea of cipherization constitutes a decisive step in the development of numeration."[109] The other great idea, of course, was that of place value, first used by the Babylonians.

The following rearrangement of hieratic numerals from problem 79 by Ahmose, trancribed into hieroglyphic numerals by Chace, illustrates the difference between hieroglyphic and hieratic numerals.



Art 16. Hieratic Numerals

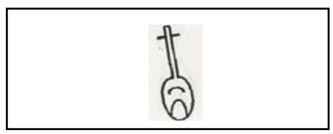
(Excerpted from *The Rhind Mathematical Papyrus*, Chace, 1929.)

Although a zero placeholder was not used or needed in Egyptian numerals, there are other applications for the zero concept. Historians such as Boyer[110] and Gillings have found examples of the use of the *zero concept* in ancient Egypt. But

Gillings wrote, “Of course zero, which had not yet been invented, was not written down by the scribe or clerk; in the papyri, a blank space indicates zero.” [111]

However, some Egyptologists knew that the ancient Egyptians did indeed have a zero symbol. This information may have been missed by historians of mathematics because the symbol did not appear in the surviving mathematical papyri. Instead the Egyptian zero symbol has been found in engineering and bookkeeping applications.

The ancient Egyptians made an interesting choice for their zero symbol. It was the triliteral symbol with consonant sounds nfr, which was also their word for beauty, goodness, or completion.[112] The hieroglyph represented the heart, lungs and windpipe, the essence of life and beauty.



Art 17. Egyptian zero

There were two major applications for the Egyptian zero symbol:

1. Zero reference level for construction guidelines

Massive stone structures such as the ancient Egyptian pyramids required deep foundations and careful leveling of the courses of stone. Horizontal leveling lines were drawn on the front face of core masonry to guide the construction of temples and pyramids. One of these lines, often at pavement level, was used as a reference and was labeled nfr, or zero. Other horizontal leveling lines were spaced 1 cubit apart and labeled as 1 cubit above nfr, 2 cubits above nfr, or 1 cubit, 2 cubits, 3 cubits, and so forth, below nfr.[113] This use of zero as a reference for measurement is similar to the use of zero in the Celsius and Fahrenheit temperature scales, for altitudes above and below sea level, and for other physical models for signed numbers.

In 1931, George Reisner described zero leveling lines at the Mycerinus (Menkure) pyramid at Giza built c. 2600 BCE. He gave the following list collected earlier by Borchardt and Petrie from their study of Old Kingdom pyramids.

nfrw	zero (Note the w suffix added to nfr for grammatical reasons.)
m tp n nfrw	zero line
hr nfrw	above zero
md hr n nfrw	below zero[114]

2. Bookkeeping, zero remainders

A bookkeeper's record from the 13th dynasty c. 1700 BCE shows a monthly balance sheet for items received and disbursed by the royal court during its travels. On subtracting total disbursements from total income, a zero remainder was left in four columns. This zero remainder was represented with the same symbol, nfr, as used for the zero reference line in construction. [115]

These practical applications of a zero symbol in ancient Egypt, a society which conventional wisdom believed did not have a zero, should encourage historians to reexamine the everyday records of ancient cultures for mathematical ideas that may have been overlooked.

Appendix C

C: Number Puzzles from Egypt

Adapted from Ahmose's Mathematical Papyrus[116]

1. Think of a number. Add 2/3 of the number. From this sum, subtract 1/3 of the sum. Tell me your answer and I'll tell you your number.

   Solution: Subtract 1/10 of answer from the answer to get your number. For example, if answer is 10, then your number is 9.

2. Think of a number. Add 2/3 of your number to your number. To this sum add 1/3 of the sum. Now take 1/3 of the result. Tell me your answer and I'll tell you your number.

   Solution: Add 7/20 of your answer to your answer. For example, if your answer is 10, then your number is 10 + (7/20)(10) or $13^1/_2$.

Legendary Problem about Diophantus

Diophantus passed one-sixth of his life in childhood, one-twelfth in youth, and one-seventh more as a bachelor. Five years after his marriage his son was born. The son died four years before his father at half his father's final age. How old was Diophantus when he died?

Answer: 84. Equation in modern form:
$1/6x + 1/12x + 1/7x + 5 + 1/2x + 4 = x$

Diophantus himself solved many number problems.

Find two square numbers such that their product added to either square number gives a square number. [**Diophantus' answer:** $(3/4)^2$, $(7/24)^2$]

Find three numbers such that their sum is a square and the sum of any pair is a square. [**Diophantus' answer:** 80, 320, 41]

Find two numbers such that their sum is equal to the sum of their cubes. [**Diophantus' answer:** 5/7, 8/7]

Find three numbers such that the product of any two added to the sum of the two is a square. [**One of Diophantus' many answers:** 1, 4, 12]

Diophantus' problems are excerpted from Howard Eves, **An Introduction to the History of Mathematics**, p. 160.

Appendix D

D: Right-Triangle Theorem

The accuracy of the right angles of the base of Khufu's pyramid, one part in 27,000, shows that the Egyptians were using mathematical principles equivalent to the right-triangle theorem: $c^2 = a^2 + b^2$. However, no papyrus has been found that explicitly states this relationship. Other clues to possible Egyptian knowledge of this principle have been given earlier in this essay. They are cubit and remen measures, and problems involving sums of squares found in the Berlin papyrus.

An ancient Babylonian tablet, known as "Plimpton 322," has some interesting columns of numbers.[117] The values show that the Babylonians knew the so-called Pythagorean theorem over 1,000 years before Pythagoras was born. The theorem continues to carry the name of Pythagoras but could be more accurately called the "Right-Triangle Theorem."

These columns contain values for "b" and "c" that satisfy the important formula of $c^2 = a^2 + b^2$. A piece of the tablet broke off and it is believed that the "a" column was lost. Students can use their calculators to find the missing column for "a" that satisfies $c^2 = a^2 + b^2$:

b	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

The answers to this exercise are shown on the following page:

a	b	c
120	119	169
3456	3367	4825
4800	4601	6649
13500	12709	18541
72	65	97
360	319	481
2700	2291	3541
960	799	1249
600	481	769
6480	4961	8161
60	45	75
2400	1679	2929
240	161	289
2700	1771	3229
90	56	106

A later Egyptian, Diophantus (c. 250 A.D.), further explored triples of numbers that satisfy $c^2 = a^2 + b^2$. Diophantus has been called the "Father of Algebra." It is believed that his work is known to us through the commentary of another Egyptian, Hypatia (died 415 A.D.).[118] It would only be just to call Hypatia the "Mother of Algebra."

Hypatia studied the Diophantine number patterns, including numbers that were the sums of squares. One problem by Diophantus reads, "Find a number that is the sum of squares, and its square is a sum of squares." A simple solution was to find prime numbers that could be generated by the formula 4n + 1, n = 1,2,3, et cetera.

For n = 1, 4n + 1 = 5 and 5 = 4 + 1, the sum of two squares. Also, $5^2 = 4^2 + 3^2$, showing that 5 units is the length of the hypotenuse in a 3, 4, 5 triangle. For n = 2, 4n + 1 is not prime, so continue to n = 3. For n = 3, 4n + 1 = 13, and 13 = 4 + 9, the sum of two squares. Also, $13^2 = 5^2 + 12^2$, showing that 13 is the length of the hypotenuse of the 5 ,12, 13 triangle. Students can make a table, using the odd values of 4n + 1 to generate values for right triangle triplets.

The Diophantine number patterns studies by Hypatia included figurate numbers. Students can use beans to manipulate these patterns. For example, to generate square numbers, start with 1, add 3, add 5, etc. Students can easily see the following:

$$1 + 3 + 5 + ... + 2n - 1 = n^2$$

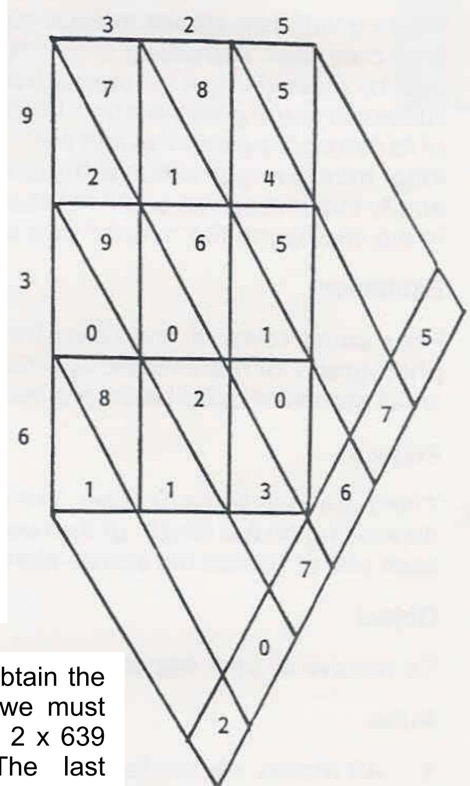
Appendix E

E: The Lattice Method of Multiplication

The lattice method of multiplication was used by Islamic scholars and introduced from North Africa into Europe during the Middle Ages. It was a big step toward the multiplication we use today, according to Al-Daffa **(The Muslim Contribution to Mathematics,** p. 40). This method was simpler than the early Hindu method.

In the example to the right, 639 is multiplied by 325. It is simpler than the modern method in that each partial product is separate, without requiring carrying until the final addition.

The multiplier 325 is listed across the top left to right. The multiplicand 639 is written on the left, starting with 6 on the bottom. Each partial product is fully written out in its respective cell. There is nothing to carry until the final diagonal addition. Compare with current method below.




```
   639
   325
  3195
 1278
1917
207675
```

In the current method to the left, to obtain the partial product of 5 x 639 = 3195, we must "carry" three times. Then we multiply 2 x 639 and carry once to get 1278. The last multiplication. by 3, requires 3 "carries" to get 1917. Of course we must carry in adding the three partial products.

Art 18. Latice Method of Multiplication

Ali-Abdullah Al-Daffa,
The Muslim Contribution to Mathematics
(Atlantic Highlands, NJ: Humanities Press, 1977), p. 40.
Used with permission.

Appendix F

F: Elementary Rules for N'tchuba

Starting with two stones in each cup, each side plays only in the two rows nearest their own side. The object is to "capture" all the stones from the other side. Moves start by picking up all the stones from one cup and then dropping one stone in each successive cup, going in a counterclockwise direction. If the last cup is not empty, all of its stones are picked up and the "sowing" continues, counterclockwise, in a closed loop, thus giving practice with "clock arithmetic." The move does not end until an empty cup is reached, and if the cup is in the inner row, any of the opponents' stones in the cup facing the "empty" cup can be captured.

Equipment

Four equal rows of cup-sized holes. These can be dug in the sand or represented by circles on a sheet of paper for playing on a table. Two small stones or dried beans are needed in each cup at the beginning of the game.

Players

There are two teams or sides. The number of players on a team can be one to five, depending on the length of the rows. The team agrees on the move to be made and each player passes the stones along his/her section.

Object

To remove all your opponents' stones while your side still has stones in holes.

Rules

1. All moves are made in a counterclockwise direction around two rows which make up a closed loop. No holes can be skipped over.

2. A move can begin from any cup on your side except that single stones cannot begin a move if your side has a cup with two or more stones available.

3. Once stones are picked up from a cup the play must be completed and cannot be retracted.

Procedure

Each team plays only on the two rows closest to it so that each side has an “inner” row in the middle of the playing field and an “outer” row next to which the players sit.

Each team alternates at first to begin a game. The team in play picks up all the stones from any of its non-empty cups. Moving counterclockwise, it drops one stone in each successive cup until the last stone is dropped. If the cup receiving this last stone was empty, the move stops.

But if this cup was not empty, the player(s) must pick up all stones now in this cup and continue, counterclockwise, to distribute a stone in each successive cup. This process continues until a last stone is placed in an empty cup. At this point it may be possible to remove stones from the opposite team. Then the other team starts its move.

Removal of Stones

If a move ends in the inner row, any stones in the facing cup of the opponents' inner row can be removed or captured. Also, any stones in the same column of the opponents' outer row may be removed.

However, had the opponents' inner cup in this column been empty, then no stone could have been taken from the outer row either. Therefore, it is important to calculate ahead, using many quick additions and subtractions.

In the complete N'tchuba, as soon as a capture is made as described above, an additional capture is permitted of stones in one other hole, any other hole on the opponents' side. This speeds the game but makes the advance calculation of the opponents' moves far more complex.

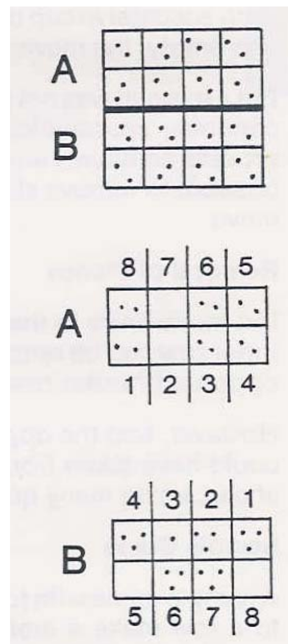
Sample Game

A sample game with four cups in a row is shown on the pages that follow. Junod says that 8, 10, 16, or 22 cups to a row make a more interesting but complex game. For the larger games, the complete N'tchuba may be preferred because it allows additional stones removal and moves faster. My teachers also told me that the game could be played with just one stone in each cup. This makes an interesting variant but I never saw it played that way.

: Sample Game of N'tchuba

As game opens
(2 in each cup)

1. Teams alternate as first to move. Let B begin by picking up two stones from cup 3. B puts one stone in 4 and one in 5. But 5 was not empty. Therefore, B picks up all three stones from 5 and puts one in 6, one in 7, and one in 8. But 8 was not empty. So, B picks up all three stones from 8 and puts one in 1, one in 2, and one in 3. Since 3 *was* empty, the move stops. B CAN NOW REMOVE OPPONENTS' STONES in the facing cup, A-2. Since A-2 was not empty, B can also take any stones that may be in the outer cup in the same column, A-7. The game now looks like Figure 1.

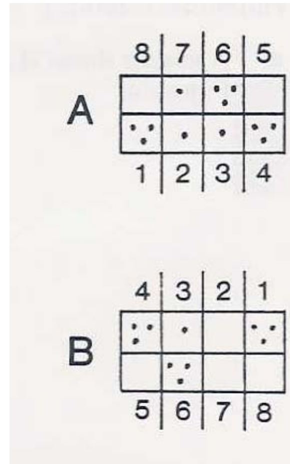


B has moved from 3
and captured A-2, A-7

Figure 1

Appendix F (cont.)

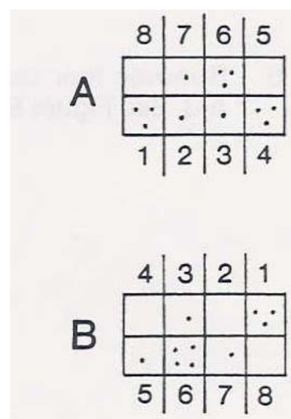
2. It is now A's turn. A would like to end in 3 for the maximum capture of six of B's stones. If A starts in 3, the clock arithmetic will bring A back to 3 because the required move will then be the same as step 1 above. Then 2 + 3 + 3 makes a complete circuit of 8. A's last stone ends in 3 and A captures three stones from B-2 and three from B-7. See Figure 2.



A has moved from 3
and captured B-2 and B-7.

Figure 2

3. B does not have a move which can end in the inner row. Therefore, B cannot capture stones but uses the move to distribute the stones away from the three stones from cup 4. At the end of B's move, the game looks like Figure 3.

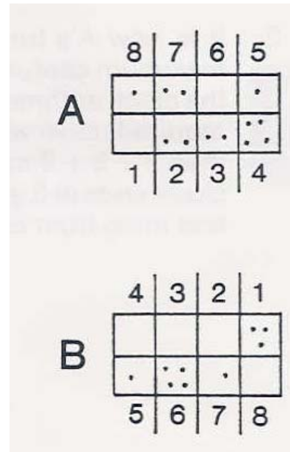


B has moved from 4.
No captures.

Figure 3

Appendix F (cont.)

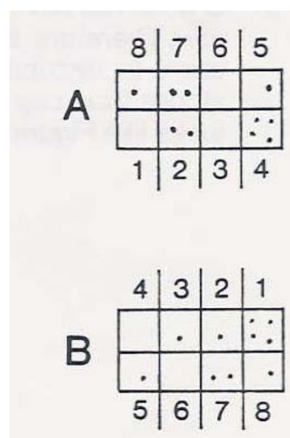
4. A moves three stones from 6 and cannot capture. See Figure 4.



A has moved from 6.
No captures.

Figure 4

5. B moves four stones from 6 and captures two from A-3. See Figure 5.

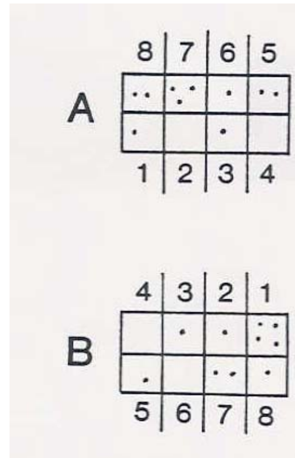


B has moved from 6
and captures A-3.

Figure 5

Appendix F (cont.)

6. A moves two stones from 2 and cannot capture because B-4 is empty. See Figure 6.



A has moved from 2.
No captures.

Figure 6

The game continues until one side has captured all of the stones.

Appendix G

G: A Game of Strategy from Mozambique

Children in Mozambique play this game of skill, which sharpens their reasoning abilities. The game can be scratched on the ground, or drawn on a sheet of paper. It is called the "Butterfly."

Nine black stones and nine white stones are used for the simple version of the game. The unoccupied point in the center is used for the first move.

Game pieces can move one place along any line, either direction, if the next place is empty. The move can be on a horizontal, or a vertical, or a diagonal line, If an enemy piece is between your piece and a free place, your piece can jump the enemy piece as in checkers. Then the enemy piece is considered "eaten" and is removed. Players alternate, starting with the light-colored stones. The game ends when the player loses all pieces. Then the player who still has a piece or pieces left on the board wins.

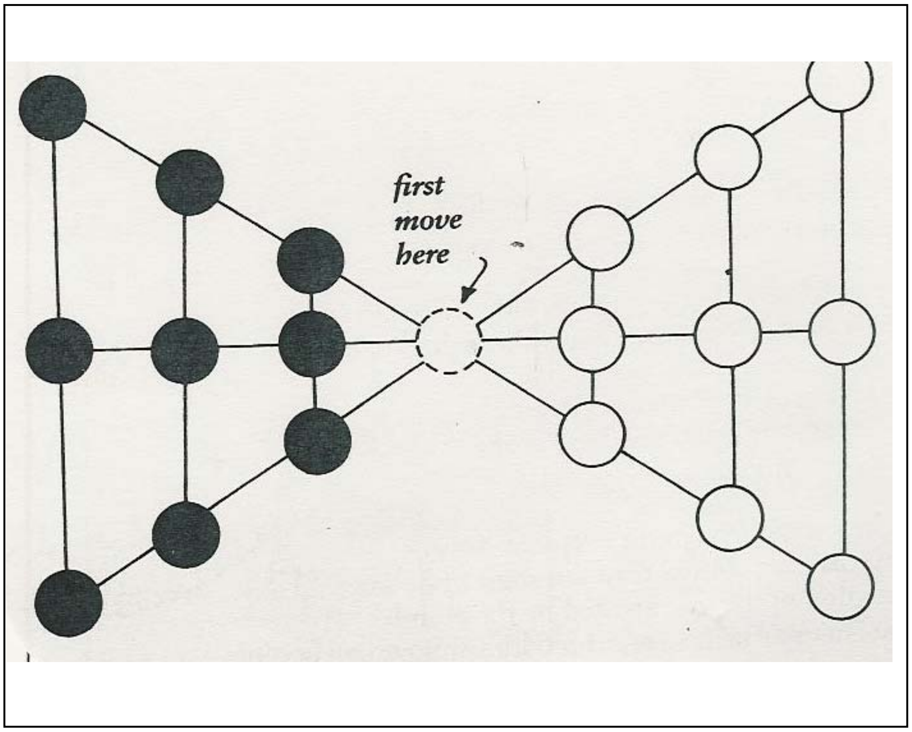
The Complete Version of the Butterfly

The complete version starts as above with each player placing 9 pieces on the board. But each player has 14 additional pieces in reserve. When a player is reduced to 3 pieces, she may place 6 more on the board, as close as possible to the 3 pieces left on the board. Each player can do this twice. Finally, when the player is down to 2 pieces, the player can place the last two reserve pieces on the board.

The next page contains a Butterfly game board.

AUTHOR: Lumpkin **SUBJECT:** Mathematics

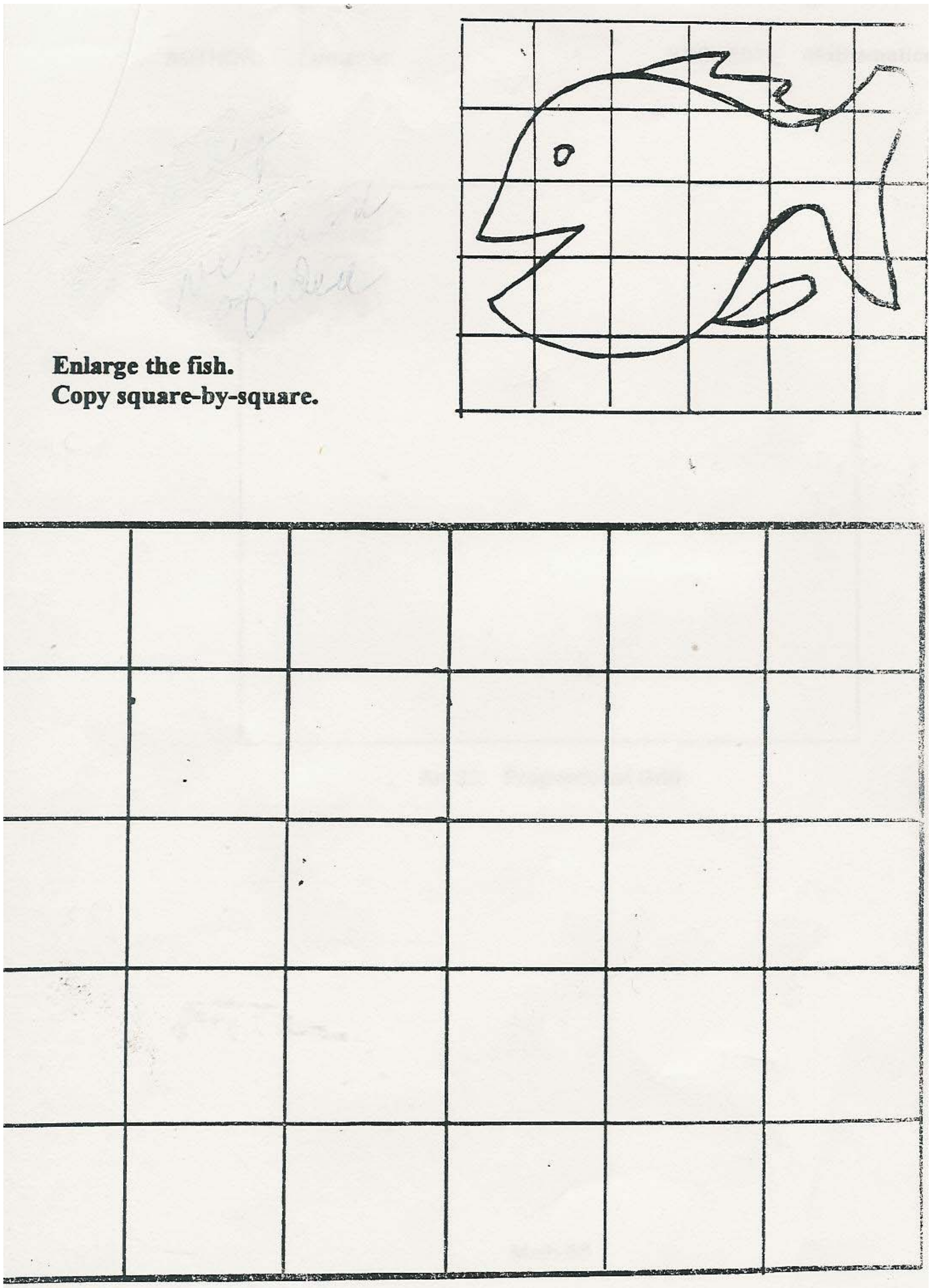
BUTTERFLY GAME BOARD



Art 19. Butterfly Game Board

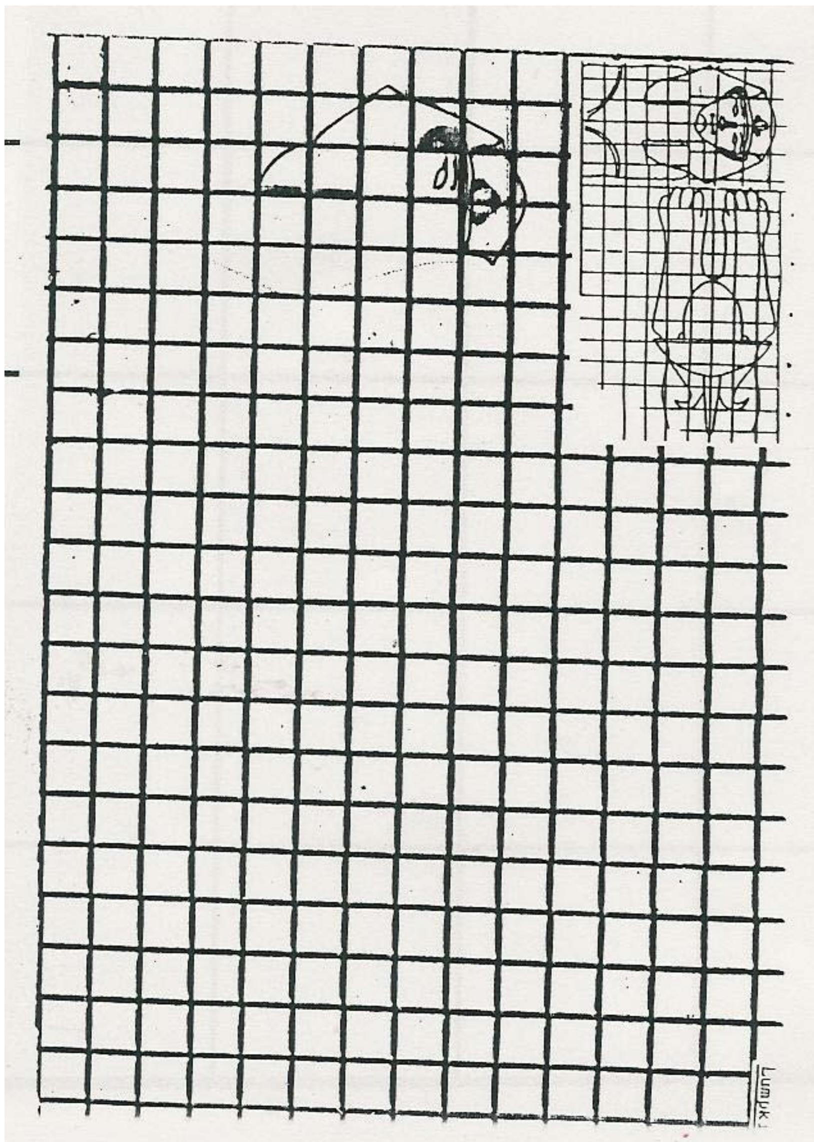
Appendix H

H: Egyptian Method of Enlarging Pictures



AUTHOR: **Lumpkin** **SUBJECT:** **Mathematics**

Using the grid on this page, copy the drawing of the Sphinx torso, square by square, preserving the proportions.

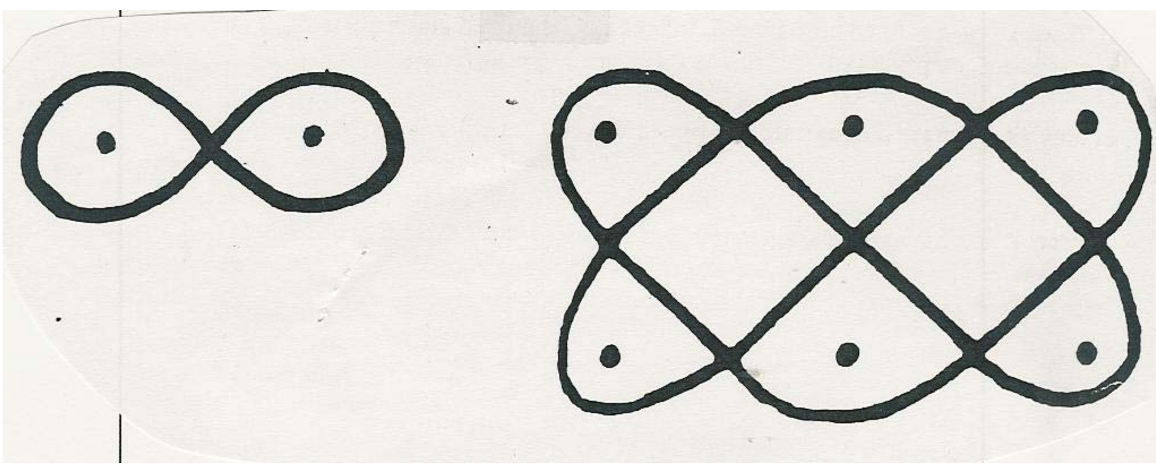


Art 20. Proportional Grid

Appendix I

I: Ethnomathematics

Ethnomathematics, a new field of study, has identified a considerable body of mathematical knowledge embodied in the traditional designs of many cultures. African network-type designs have been analyzed by Professor Paulus Gerdes, chair of the African Mathematical Union Commission on the History of Mathematics in Africa (AMUCHA). His book, *Lusona: Geometrical Recreations of Africa*, contains designs that can be expanded to form a series of drawings, as shown in the following example:



Art 21. Geometrical Recreation

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