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American Indian Mathematics Traditions and Contributions

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Biographical Sketch of Author

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AMERICAN INDIAN MATHEMATICS
TRADITIONS AND CONTRIBUTIONS

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INTRODUCTION

American Indians generally have had a pragmatic orientation to the use and study of mathematics. In most Indian cultures, mathematics traditionally was practiced by most of our ancestors, when and if they used it, for its value in daily life rather than for its own sake or as an intellectual challenge. Counts and record-keeping relating to economic activities have long been the major use for mathematics among most American Indian cultures, so far as ethnologists and other specialists have been able to learn.

American Indians did integrate mathematics into other areas of daily life. As is true of most cultures, we found the principles of this discipline broadly useful as a way to think about and describe many parts of human experience, especially those with quantitative and relational aspects. Our native ancestors also developed an esthetic regard for what their counts and calculations revealed about order and pattern in the world. Certain American Indian cultures well appreciated the usefulness and beauty of mathematics in activities related to history, engineering, architecture, astronomy, calendrics and the religious practices associated with these fields.

However, accessible written records of American Indian techniques in mathematics are extremely rare. Also rare are other forms of Indian records in this field of human knowledge. This is due to a number of causes.

Many American Indian cultures did not record their knowledge, or recorded only parts of it, in the visual form of a system of writing or symbols. Traditionally, our ancestors preferred to rely on the intimacy and interactive characteristics of the oral tradition as a teaching and storage medium for what they knew of numbers, order and pattern. When a great many elders died prematurely during the epidemics and wars that were part of the conquest of the Americas, we lost a significant portion of our mathematical traditions. Modern Indians and others wishing to know the American Indian traditions in mathematics have encountered many gaps in what knowledge remains in the oral tradition.

Indian and non-Indian scholars alike are also still learning how to decipher the surviving 'written' materials pertaining to American Indian mathematics. Very recent advances have occurred in the translation of the Maya glyphs, to cite a major example. Great numbers of these symbols were carved onto monuments or were painted into books, a handful of which survived the Conquest. Mathematical content is a significant part of the matters recorded in this most advanced of American Indian writing systems. These new translations are leading to increased knowledge of Mayan thought and mathematics, and scholars may soon be able to make this information more generally accessible.

Reports of 16th and 17th century Spanish scholars and surviving pre-Contact documents are evidence that a great deal of recorded Indian knowledge, some of it pertaining to mathematics, was lost in the destruction of Mesoamerican and Incan libraries in those centuries. This tragedy resulted from the misguided attempts of colonial religious, military and civil authorities to erase the native cultures of the Indian nations that came under Spanish domination in that era.

Thus, what remains today of traditional American Indian mathematics are fragments of the living oral tradition and a very few surviving texts. Scholars can supplement these remains with the accounts of some early European observers and those of a few bilingual Indian historians from the period shortly after Contact. There is also a recent body of knowledge. This comes from scientific reconstruction of Indian mathematics deduced from art works, buildings, monuments, engineering works and other archaeological remnants. Each source has its limitations for purposes of general study.

We can read parts of some of the native texts while others have yet to be translated. Much of the remaining oral tradition is only shared among family, clan and tribal members and is not available for consideration outside those circles. Some parts of the oral tradition and Indian literary sources have been transcribed into modern books; while some of these can be helpful, others contain errors, misunderstandings and misinformation. Many contemporary Indians who know something of the traditions caution that the materials in the books can't always be taken as accurate accounts of how things were done or understood within the culture. Unfortunately, it is difficult to sort these matters out without significant 'insider' knowledge from the particular culture(s) involved.

The accounts of explorers and colonial officials generally share these limitations. As concerns Indian mathematics, few contain much clear and useful information. (Most of the accounts I have examined over the years reveal the authors' lack of attention to or incomprehension of Indian mathematical techniques. Actually, only a few explorers or colonial writers give much evidence of familiarity with mathematics in general.) The works of Indian historians, like Guamán Poma de Ayala of the Inca or the numerous Mexica ('Aztec') collaborators of Father Bernhardino de Sahagún, give a better picture of Indian achievements. Yet even they discuss mathematics as a minor matter compared to other concerns of their time.

The interpretations and reconstructions of modern scientific researchers are the most accessible materials from which a reader can gather some sense of American Indian mathematics. Here one must keep in mind the shortcomings of the interpretive methods of one culture when applied to fragmentary materials from another.

Traditional Indian mathematics is thus a discipline in which much of the original richness is lost, much remains to be relearned, and much which we think we understand may have to be revised as new insights become available. What is offered below is an introduction to some of the better understood Indian mathematical achievements for teachers who wish to offer their students access to this aspect of American Indian cultures.

The author gratefully acknowledges the debt owed here to Michael P. Closs of the University of Ottawa and the other researchers who contributed to **Native American Mathematics**. This is the foremost extensive treatment of the subject currently available in English to teachers and students of Indian mathematics.[1]

NUMERATION SYSTEMS

Numeration and Counting

Numeration refers to the naming of numbers. Counting is the use of named numbers in a one-to-one correspondence with selected objects to determine the quantity of such objects. A numeration system is a set of rules for establishing the names of numbers. These rules often include the set of symbols conventionally used to represent the names of numbers. They may also define the ways, such as place value, in which to group the symbols to express larger numbers.

Michael Closs, in his essay “Native American Number Systems,” reports that Indian tribes in much of North America use or traditionally used base-10 (or ‘decimal’) numeration systems.[2] Among North American linguistic groups using decimal numeration are the Salish, Algonkin, Siouan, Athabaskan and Iroquoian –speaking tribes. In South America, the Quechua-speaking cultures (the best known of which was the Inca) also use decimal numeration. However, there have been many alternatives to a decimal numeration system used throughout the Americas. This fact leads Closs to see in the mathematics traditions of the diverse Indian tribes of the Americas "...a cultural mosaic in which independent invention could and apparently did flourish."[3] In his essay, he offers examples and counting terms from more than twenty American Indian cultures to illustrate some of this diversity.

Closs notes that the major American Indian alternative to a decimal numeration system is the base-20 or 'vigesimal' system. This was used by most of the advanced civilizations of Mexico and Central America. Some California tribes and the Caddoan-speaking tribes of the south-central U.S. also developed vigesimal systems. William Folan, looking at records made by explorers and later ethnologists among the Nootka people of Vancouver Island in what is now British Columbia, affirms that the Nootka system of numeration is vigesimal.[4]

There are still other numeration systems created by American Indians. One is a mixed base-8/base-16 system employed by one group of the Yuki of Round Valley in northern California. Apparently this derived from counting with the eight spaces between the fingers, rather than the digits themselves. The counting names for units go up to sixteen before recombining and repeating to give higher numbers.[5] Similarly, the Chumash tribes of coastal southern California used a mixed base-4/base-16 numeration system.[6]

The Bororo and Bakairí cultures of the Matto Grosso region of Brazil use a base-2 numeration system, forming higher terms by repetitions and addition of the two base numbers.[7]

The foundation for traditional Inuit ('Eskimo') base-5/base-20 counting appears to be the digits on one hand, then the next hand using a repetition of names for the first five digits plus an auxiliary term, and so on until the 20 digits of a human body are counted. Modern Inuit counting has been modified to make trade contact with

European–Americans easier, and is effectively a base-10 system.[8]

A few American Indian cultures use no base at all in their numeration systems.[9] A baseless system is one in which words are not repeated, modified or combined to create new terms for counting higher numbers. Examples of this lack of a base system are the Siriona tribe of Bolivia and the Yanomamo of Brazil who have counting words for only the numbers ‘one’, ‘two’ and ‘three’. Beyond this, words exist meaning ‘much’ or ‘many’; some of the higher references are made specific by holding up digits of the hands.[10]

The lack of a name for specific larger numbers does not necessarily indicate that members of a tribe were unable to count or refer to such numbers, as Maurizio Gnerre has pointed out. Gnerre cites examples of higher Jívaro counting using gestures based on digits of the hands and feet, even though the Shuar language spoken by the Jívaro does not have words designating numbers larger than five.[11]

Counting by American Indians in either decimal or vigesimal systems, according to Closs, is generally based on one-to-one correspondence between digits of the human body and objects being counted. A decimal system usually based its terms for counting numbers on the ten digits of the hands, while a vigesimal system often also included terms based on the digits of the feet to make up the base of twenty. As mentioned above, Closs gives specific examples of counting words (and their digital referents) from several American Indian and Inuit languages in his book and in a separate essay entitled “Mathematical Development in the New World”.[12]

Many tribes took another route to form the counting numbers, however. Closs mentions a study of over 300 American Indian languages done by William Eells in the 19th century. Eells found about 40 percent of the tribes he studied formed numbers below the base limit of ten or twenty by arithmetic operations, much the way speakers of

Spanish or German form numbers higher than twenty in their languages.

Eells documented frequent use of addition and multiplication, with lesser use of subtraction, in forming the terms for some of the smaller counting numbers. He even noted two cases, in the Pawnee and Unalit (an Inuit group) languages, where division is used to create the counting numbers of 5 and 10, respectively.[13]

American Indian counting systems vary in the upper extent of numbers that can be counted or named. Closs cites examples from the 19th century historian and Indian Agent Henry Rowe Schoolcraft, whose monumental (if sometimes flawed) studies of Indian cultures documented many tribes counting to around 1,000 or more.

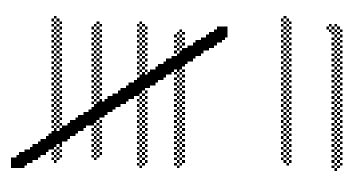
The Apache and Choctaw have counting terms up into the hundreds of thousands. Such diverse cultures as the Ojibwe, Lakota, Wyandot, Cherokee, Winnebago and Micmac are able to count into the millions and even billions using their numeration systems. Surviving Precontact documents and historians' accounts from the early Conquest era show that the Inca, Aztec and Maya mathematicians could count and calculate with numbers in the millions with no apparent upper limit.[14]

Closs states that number systems such as these, which used the principles of grouping and arithmetic operations, could allow American Indian mathematicians to precisely quantify numbers "equivalent to the set of positive integers" and are thus "mature number systems."[15]

Numerals

We call the visual representation of a number in the form of a symbol a numeral. The spoken representation of a number in the form of a word is also a numeral. In either case the numeral is a distinct symbol representing a particular quantity, a number. Numerals allow us to communicate about numbers in ways that go beyond gestures. Of particular value is the visual form of a symbol for a number, which allows us to record that number on media that can outlast one or many human lifetimes. Such records, where they survive the passage of time, give us the opportunity to see something of our ancestors' mathematical knowledge.

A very basic and early form of numeral is the tally, a mark made on rock, wood, bone, hide or some other surface by pecking, incising or painting. Tally marks have been used by cultures all over the world, including many in the Americas. Tally marks usually have a one-to-one correspondence to some counted object. Some marks in a tally system may have additional meaning indicated by their positioning. A familiar modern example is shown below, where the diagonal tally not only counts as 'one', but is a visual cue that a group of five has been completed.



William Murray has studied rock art sites in the northern Mexican desert of the Coahuila Plateau near Monterrey. Evidence of human habitation in the area goes back about 10,000 years to the beginning of the current inter-glacial period. At a site called Presa de La Mula he studied ancient pecked dot and grid marks in carefully arranged series that he believes are tallies. His analysis, still controversial, suggests that the grid and tally marks are observational records that were used to determine the length of lunar months.

He notes that the tallies, thought to have been made by hunter-gatherer bands in the region, include several distinctive 'correction' symbols. Murray's reconstruction of the system suggests that the count of days and grouping the count into lunar months had to be reconciled about three times over a seven-month period by the observers before the count system approximated actual lunar cycles with fair accuracy.[16]

Murray has also interpreted rock art tallies at several other early sites in the region as remnants of early Indian astronomical observations and records of hunting successes.[17] He believes that knowledge of basic cycles of time probably helped the early hunters become more effective in taking seasonal and migratory game.

Michael Closs collected reports of a number of uses of tally marks among the Ojibwe of the northern Great Lakes region.[18] In northern Minnesota, Ojibwe village members carved wooden census records, using a pictographic totem symbol to represent each family. They carved tally marks next to each totem symbol to show the number of members of that family.

Closs notes that a clan chief once showed an old incised copper plate with tallies indicating the number of generations that had passed since his clan took possession of its territory. He notes that the American historian Henry Schoolcraft (who was married to an Ojibwe woman for a time) reported that Ojibwe grave markers formerly used tally marks to indicate the number of certain kinds of important events that had occurred in the deceased individual's life. Closs adds that the ethnologist Frances Densmore wrote in the 1920s that Nodines, an Ojibwe elder, once told her about the use by her father of daily tally marks on a 'year-stick' he kept as a calendar.

Dr. Closs also reports that some of the pictographic records kept on birch bark scrolls by the Ojibwe Midéwewin medicine lodges include extensive use of tally marks as numerical records.[19] He draws examples from a 1975 analysis of 137 Ojibwe scrolls by Selwyn Dewdney. One important scroll, illustrated on page 202 of Closs' book, depicts the four grades of shaman in the Earth Midéwewin as lodges. The first three lodges show the number of shamans in each by means of pictographic representations of human figures. In successive order, these lodges have 4, 8, and 16 members (the number four and its multiples have profound spiritual significance in many American

Indian societies). The fourth and highest grade shows the use of tally marks instead of human figures to denote the 32 shamans in this grade of the Earth Midéwewin lodges.

Mayan Numerals

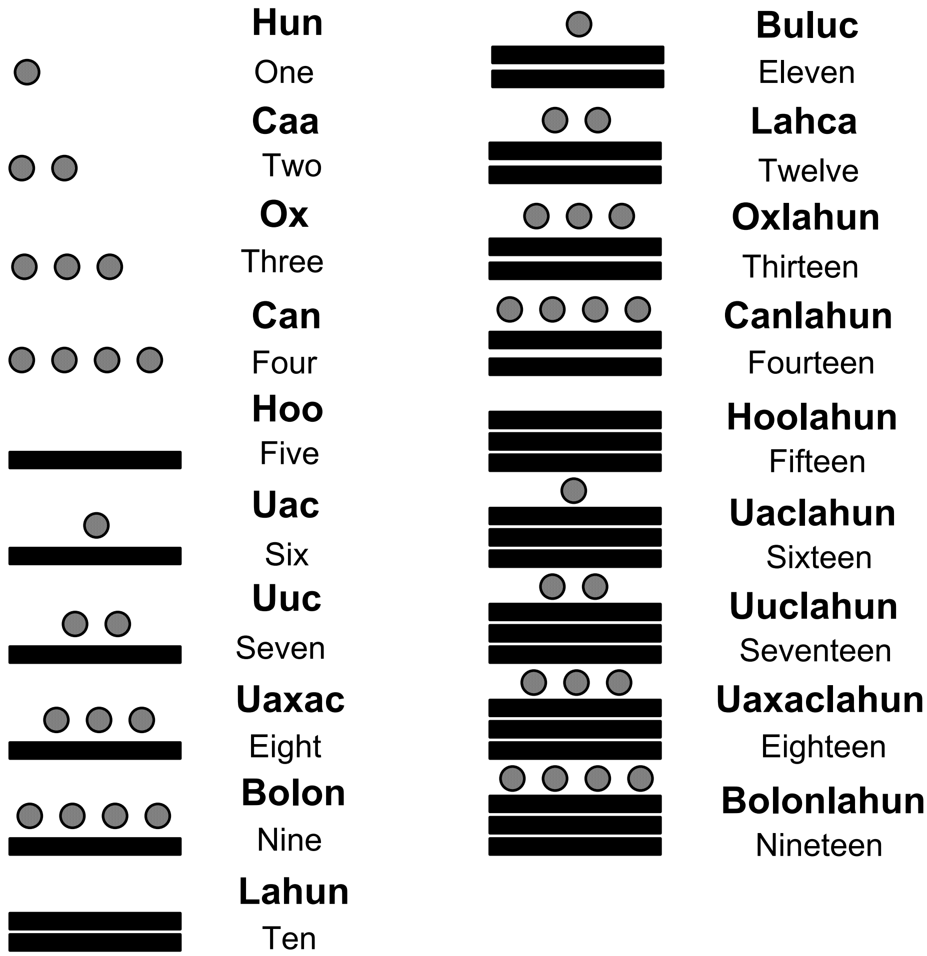
The Maya culture reached a high level of sophistication in mathematics during its Classic Era, around 200 – 900 A.D. The Maya city-states were then the dominant culture in what are now the modern-day regions of southern Mexico, Guatemala, Belize and parts of Honduras and El Salvador. Their mathematicians and priests (frequently the same persons) used a vigesimal (base-20) number system and wrote their numerals in two main ways, with a third way that combined the two main systems.

'Bar and dot' notation was the most common notation for numerals throughout much of Mesoamerica. The Olmec and Zapotec cultures both used it in a form that preceded the Mayan version. The Maya commonly used it to represent numbers in contexts outside their calendar system.

Bar and dot notation somewhat resembles the simpler tally mark system in that only three types of marks are needed and all the numerals represented are just unique combinations of these three marks. A dot stands for one unit while a bar stands for five units. The third symbol, that for zero, is discussed below. Bar and dot notation can be written either horizontally or vertically with respect to the alignment of the bars.[20] The illustration on the next page shows the horizontal style for the numbers from one to nineteen.

If the numbers are written with the bars vertical, the dots are conventionally placed to the left of the bars.

Numbers larger than nineteen are symbolized in the bar and dot system by using a positional notation. This worked by stacking groups of bars and dots vertically in ascending powers of twenty.[21] An important expansion to the utility of this positional system came about when the Maya developed a symbol for the concept of zero.[22]



Suggestions for pronouncing these Yucatec Mayan names for the numbers:

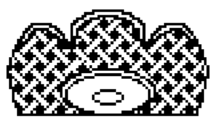
- double vowels are spoken with a glottal stop
- X is spoken as " sh "
- C is spoken as " k "
- the vowel U at the start of a word is spoken as " w "
- vowels are spoken in soft form

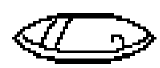
Without a numeral for zero, it would be hard to know whether no entry in the units position of a bar and dot stack meant ‘zero’ or the position simply wasn’t there, allowing the reader to mistake the 20^1 position for the units position.

The Maya were the second culture known to have developed both the idea of zero and a numeral to symbolize it. The Neo–Babylonians of Mesopotamia first developed the idea and a notation for zero by around 300 B.C. But their base-60 positional notation system only used this symbol in internal positions of a numeral and not at the units position of a numeral.[23]

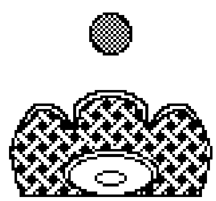
Archaeological evidence suggests that the Maya mathematicians began to employ the zero in a more modern fashion, including its use in the units position, about two thousand years ago. The Hindus would not develop their comparable version of the concept of zero or a symbol for it until several hundred years after the Maya, sometime between the 6th and 9th centuries A.D.[24]

The Mayan symbol for zero is now sometimes called the 'shell design.' The Maya refer to it as lub (pronounced 'loob').[25] Its two essential forms (there are variants) are an earlier three-lobed shape and an elliptical shape, usually with three or four distinctive internal markings. The latter was used as the symbol for lub in many of the later Mayan writings; the shell design was the earlier version. Examples of each are shown below:

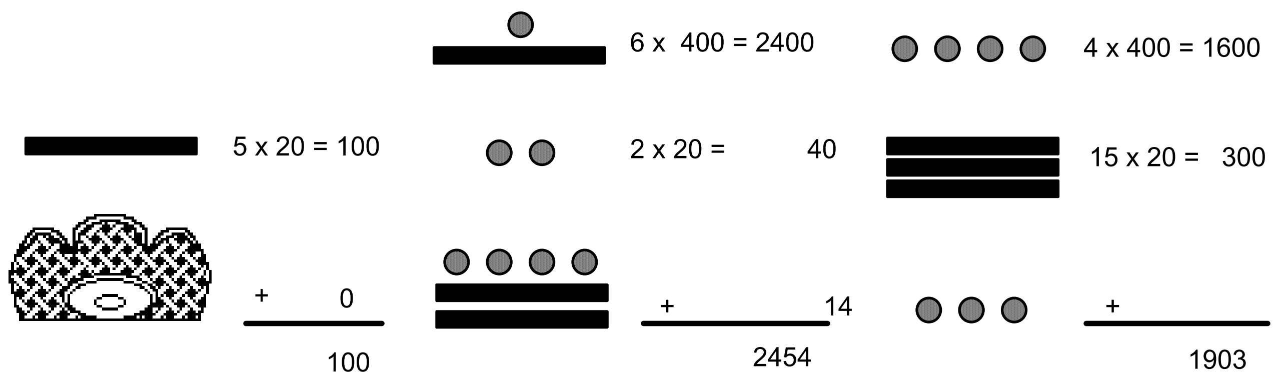




With such a zero symbol, the bar and dot numeral for twenty could be vertically written:



Since the (upper) position for the first power of twenty shows one dot, there is one twenty at that position; the 'shell design' in the (lower) units position shows there are zero units in this numeral. Several other examples of numerals in the bar and dot notation are shown on the next page with their Hindu-Arabic equivalents.



For a native speaker of English, naming the higher numbers in the Mayan vigesimal system may at first seem slightly complex compared to doing the same thing in decimal-based English, but it is really no more complicated than in many languages. According to Michael Closs, the Maya term for ‘twenty’ is spoken in various dialects as kal, uinic, or may. Counting ‘hun kal’ in the Yucatec dialect means ‘one score’ or ‘one twenty’; ‘caa kal’ means ‘two score’ or forty; ‘ox cal’ refers to ‘three score’ or sixty and so on.[26]

Numbers between a multiple of twenty in the Mayan vigesimal system are named by combining terms for a multiple of twenty with one of the terms given above for the numeral between one and nineteen that corresponds to the intervening portion of the number. There were two ways in which this was done, according to Dr. Closs:

> In the first system, prevalent in many Mayan languages today, the intervening quantity was named and placed in the ordinal-numbered score or other power of twenty to which it belonged. The second method of expressing compound numerals was to use a conjunction as we do, either expressed (catac in Yucatec) or implied by juxtaposition of two orders of components, and proceeding from the higher-order to the lower-order components. Thus, for example, 51 could be either buluc tu yox cal, ‘eleven in the third score’, or ca kal catac buluc, ‘two score and eleven’.[27]

As a matter of interest to those who might like a sense of the power of Mayan numeration, the names for the first few powers of twenty (corresponding to our decimal system of tens, hundreds, thousands and so on) are given by Dr. Closs in the Yucatec

Maya dialect as: 20^1= kal, 20^2= bak, 20^3= pic, 20^4= calab, 20^5= kinchil, and 20^6= alau. [28] Using just these six names in combination with the units already named (and following one or the other of the methods indicated in the paragraph quoted above), you can name any numeral up to 64,999,999 in the Yucatec dialect of Maya.

The Maya also used glyphs as a way to symbolize many of their numbers. This was the second of the main ways the Maya wrote numerals. Again, the system is vigesimal, so there are glyphs for the numbers from one to nineteen, plus lub, the 'shell design' for the zero. There is also a glyph for kal or twenty.[29]

The first thirteen numerals in the Maya glyph system are stylized depictions of the heads of the gods embodying the numbers from one to thirteen. Ten is the head of Death, and the lower jaw of Death is appended to the glyphs for the numbers from four through nine to form the glyphs for the numbers fourteen through nineteen.[30] Representations of these glyphs appear on the next page.

Knotted Cord Numerals

Another form for symbolizing numerals was the knotted cord. The Nootka used knotted cords and bundles of sticks as mnemonic devices to help them keep track of numbers. The cord devices reportedly kept track of a number of events, or animals killed, or the days spent in certain rituals. The stick bundles recorded numbers of high-ranking guests invited to potlatch ceremonies.[31] In both cases, knots or sticks seem to have stood in a one-to-one correspondence with the item enumerated.

After an illustration by Closs in Michael Closs,**Native American Mathematics.**
Austin: University of Texas Press, 1986. Used by permission.

The Quipu

The most famous of the knotted cord numeral systems was developed in South America. Used and probably developed by the Inca, it has become known to the world as the quipu. Scholars are not sure when it was developed, but most seem to think it did not predate the rise of the Inca state. Thus, the quipu system probably developed sometime after about 1100 – 1200 A.D. as the Inca began to expand their control beyond Cuzco, their capitol. The quipu was also eventually used by the Chibcha of Bolivia.

The scholar Bertrand Flornoy gave this description of the quipu:

> The Incan quipou [sic] was principally made of a greyish-white rope which was twisted between two thinner cords. From this rope hung 48 secondary cords, divided into five groups. To some of these cords were affixed extra threads. There were in all 87 cords...Knots were made on each cord, starting from the lower end...The first series of knots [on a particular cord] represented units, the second tens and the third hundreds....There were cords of several colors whose position [and color] indicated certain meanings.[32]

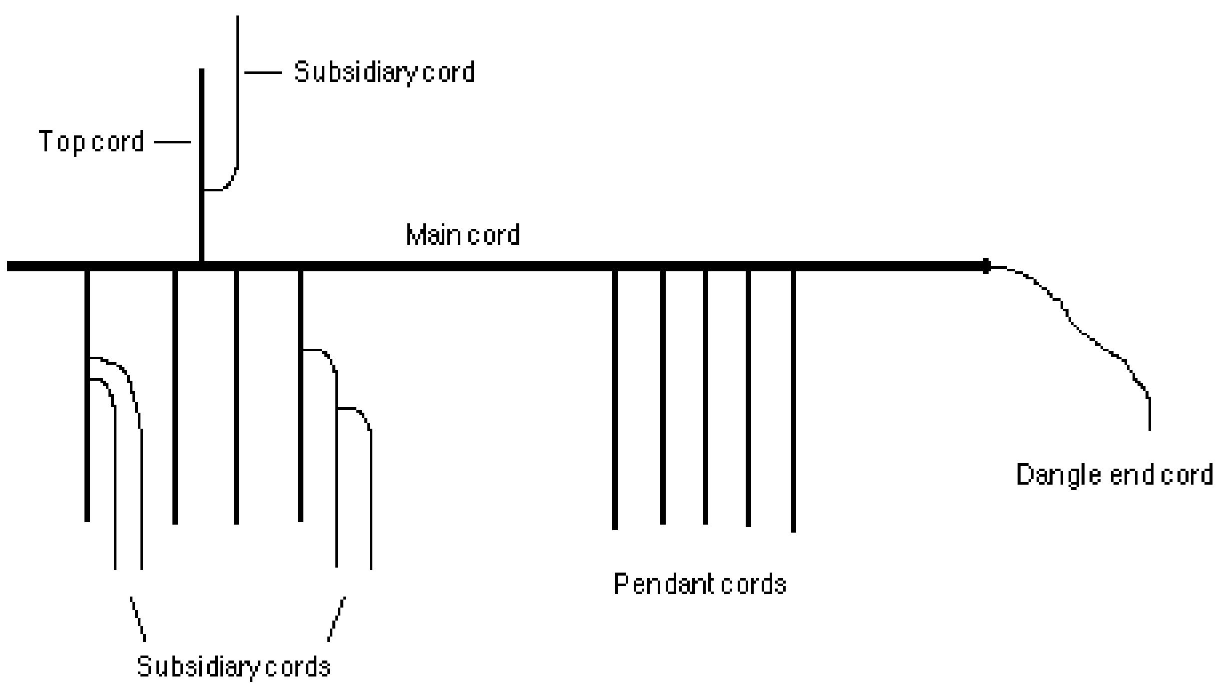
Quipus were not quite as uniform in their structure as Flornoy suggests in the quotation above. The scholars Marcia and Robert Ascher completed a study of nearly 450 of the known surviving quipus in the late 1970s.[33] They prepared technical descriptions, drawings and photographs of 190 of these as an aid to further study by others. Some of these photographic examples appear in a chapter by Marcia Ascher in Michael Closs' **Native American Mathematics.**[34]

Ascher's summary description of the structure of quipus is worth citing at length:

> In general, a quipu has one cord, called the main cord, which is thicker than the rest and from which other cords are suspended. When the main cord is laid horizontally on a flat table, most of the suspended cords fall in one direction ("downward"). These are called pendant cords. Sometimes, some of the suspended cords fall in the other direction ("upward") and so are called top cords. Suspended from some or all of the pendant or top

> cords are other cords called subsidiary cords, These can have cords suspended from them so that there can be subsidiaries of subsidiaries and subsidiaries of them and so on. Sometimes there is a single cord attached to the end of the main cord. Since the way it is attached is different than the attachment of a pendant or top cord, it is referred to as a dangle end cord. All attachments are tight so that cord positions relative to each other are fixed. Larger spaces between some adjacent cords sometimes set off groups of cords from each other. Pendant cords, top cords, and subsidiary cords are about 20 to 50 cm long. A quipu can be made up of as few as three cords or as many as two thousand cords and can have some or all of the types of cords described. A schematic of a quipu is shown [below].[35]

This illustration of the components of a quipu may help as you reread Asher's description above.



After an illustration by Ascher in "Mathematical Ideas of the Incas" in Michael Closs, **Native American Mathematics.** Austin: University of Texas Press, 1986. Used by permission.

Ascher, like Flornoy, states that color coding was used on quipus to denote associations or differences among cords in a group or across groups. In the book she wrote with her husband Robert in 1981, **Code of the Quipu**, Marcia Ascher states that the colors used in the coding system ran into the hundreds.[36] She provides examples showing how quipus were structured by combining physical groupings of cords, varying the numbers of cords in a given group, varying color patterns which could be repeated within or across groups, and attachment of (often multiple layers of) subsidiaries.

Interrelated knotting and grouping patterns, combinations of related strings and color codes enabled the quipu to serve as a rich medium for the formation of arrays and hierarchical structures functionally equivalent to modern 'tree diagrams'.[37]

Ascher's studies of surviving quipus also indicate that the 'logical structure' of a quipu's 'tree diagram' was prepared before entering any 'data' in the form of carefully positioned knots on various cords. She explicitly compares the preparation of a 'blank' quipu to the creation of a blank matrix table before a modern mathematician enters any of the elements. Ascher writes,

> ...It is, always, the structure which is provided for the data, rather than the data itself, that carries the logic of the relationship between the data items. And, of course, data is only meaningful when viewed within the context of its logical structure.[38]

The "data itself" was entered on the quipu cords using three styles of knots. Recall that the Inca used a base-10 number system. They symbolized their system of place value notation by tying groups of knots on the quipus. Starting from 'ones' at the free end of a cord, each successive group of knots on the cord (as one moves 'higher' toward the attached end) represents the next higher power of ten. The first group, representing 'ones,' would be tied with a 'long knot' (commonly called a 'fisherman's knot') showing two to nine turns with the numeral's value indicated by the number of turns. 'One' was indicated with an alternative, figure-eight knot in this first position on the cord, since long knots require more than a single turn to tie.

Numerals expressing units in higher powers of ten, further up the string, would be tied in groups of one to nine overhand knots. According to Ascher, the highest number observed on any of the surviving quipu strings is 97,357. In place value terms, five 'powers of ten' positions were needed for the groups of knots representing this number on its cord.[39]

The Aschers' study of quipu strings also showed that the Inca could represent more than one number on a given cord. This was made possible by the fact that there were distinctive knot styles (the figure-eight and 'long' knots) for designating integers with unit values from one to nine. Essentially, if one of these styles of knot appeared further up a cord from an earlier series of knots, it signaled that the positional values from that point up were to be reset starting from units. Marcia Ascher reported in her chapter in Michael Closs' book that examples of up to three separate numbers on a single cord are known from the surviving quipu strings.[40]

The quipu system had a way to represent the concept of zero, even though there was no special 'symbol' or knot for this value. Since the place-value alignment of knots was kept fairly constant from cord to cord, a position without a knot could be recognized as showing the value of 'zero' for the particular multiple of ten associated with the position, when one cord was compared with others.[41]

The quipu could also store numbers used as labels for some object or relationship. Ascher compares this use to our modern way of identifying a person by a Social Security System number or of specifying a particular commercial product through use of a part number or product code. When such number labels are combined with the contextual system of the quipu's positional structure and color coding, the resulting quipu, according to Ascher,

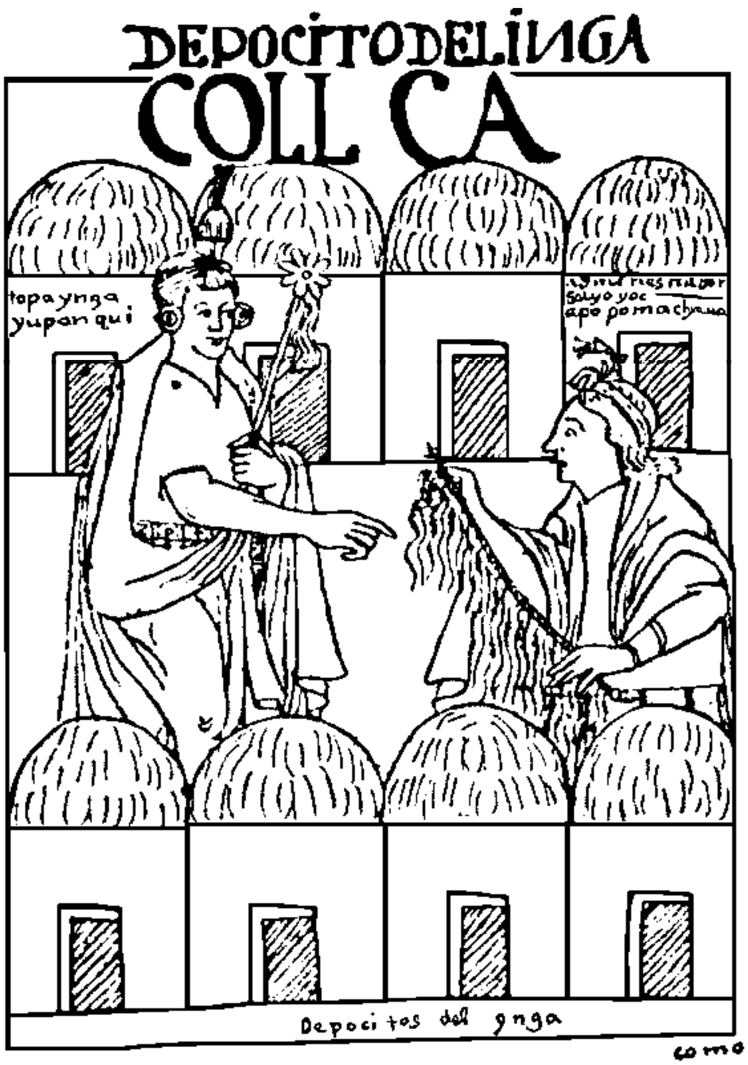
> ...is a far more general recording device than was previously believed. Numbers and number labels combined with logical structures make the

> quipu considerably more complex and sophisticated than the [one-to-one correspondence] knot records or mnemonic devices with which they are so often misassociated in mathematical literature. They differ from these also in that they constitute the sole recording system used by a state which carefully planned and executed large projects involving both people and natural resources. While quipu construction and interpretation was limited to a special but important class of people in the Inca Empire, quipus were a universal system rather than a personal *ad hoc* device.[42]

The Aschers offer a number of examples and exercises to help develop understanding of some of the techniques of quipu coding. Teachers and students can find these in chapters 5 through 7 of their book **Code of the Quipu.**[43]

The Spanish destroyed the extensive Inca libraries of quipu records in the late 16th century, soon after the Inca state's military resistance broke down in 1572.[44] About 550 known quipus survive today, scattered among museums around the world.[45] The bilingual Inca historian Guamán Poma consulted some of these remnants and the surviving Inca librarians in preparing his book **New Chronicle and Good Government** in the late 16th and early 17th centuries. Guamán Poma himself claimed to be a descendant of one of these specialists.

Our present-day knowledge of quipus derives in part from his description of the system and its uses. One of his illustrations, reproduced on the following page, shows a curaca, or accountant, reading off a quipu record the contents of a royal warehouse at the request of Topa Inca Upanqui, the tenth Sapa Inca or 'royal' Inca.



In this illustration by the Inca historian Guaman Poma, the tenth royal Inca Topa Inca Yupanqui arrives at a collca or warehouse complex and demands an account of the goods in storage.
The curaca (accountant) produces the quipu string record of the collca and reads off the contents and their amounts.

From an illustration in Burr Brundage, **Lords of Cuzco.**
Norman: University of Oklahoma Press, 1967. Used by permission.

A Numeral System Used in the Valley of Mexico

Herbert Harvey and Barbara Williams uncovered evidence of a specialized numeral system used in the province of Texcoco (in the Aztec-allied kingdom of Alcohuacan). They found a pair of census/tax documents prepared by Texcocan Indians in the first generation after the conquest of Mexico.

Unlike the word-based number references that predominate in most Nahuatl documents (see discussion below in the section on Arithmetic and Computational Systems), these documents used a distinctive system of line and dot symbols with a form of positional notation and a special symbol for zero. The Texcocans evidently developed (or borrowed from the Maya) the idea of a true zero.[46]

In this system, a vertical line represented one unit, a ‘bundle’ of three vertical lines surmounted by an inverted ‘U’ shape denoted a group of five units, a dot represented twenty units, and a glyph of an ear of corn, the cintli, symbolized zero.[47] A discussion of how this Texcocan system was used in computations of taxes on land areas appears in the following section.

ARITHMETIC AND COMPUTATIONAL SYSTEMS

We have very little concrete knowledge of the computational techniques used by most American Indian cultures in Precontact times. From the structure of various Indian languages’ words for numbers, as discussed above, we can infer that most American Indian cultures understood and practiced the four basic arithmetical operations of addition, subtraction, multiplication and division.

Linguistic analysis of number terms in Nahuatl, the language spoken by the Toltec, Aztec and numerous other civilizations in highland Mexico, shows that they used a base-20 numeration system. Higher numbers are named by terms that show that the Nahuatl-speakers used a factorial process based on powers of 20 in their conception of number structure. Stanley Payne and Michael Closs give as an example:

113,197 = 14X8000 + 2X400 + 19X20 +17 or, in Nahuatl:

= matlactli onnauhxiquipilli ipan ometzontli ipan caxtolli onnauhpoalli on caxtolli omome[48]

The Nahuatl expression, taken by its factors, translates into English as

matlactli onnahui – ‘fourteen’ **xiquipilli** – ‘eight thousands’ [20^3 = 8000]

ipan – ‘on top of’ or ‘plus, when between multiples of 20’

ome – 'two' **tzontli** – 'four hundreds' [$20^2 = 400$]

ipan – 'on top of' or 'plus, when between multiples of 20'

caxtolli onnahui – 'fifteen' **poalli** – 'twenties'

on – 'plus, when within one multiple of 20'

caxtolli – 'fifteen' **om**('plus, when within one multiple of 20')**ome** – 'two'[49]

Obviously, this is a fairly sophisticated way of conceiving of integers (if a little difficult at first for those of us accustomed to base-10 counting). It suggests that educated Toltec, Aztec and other Nahuatl-speaking peoples could readily add and multiply any integers, since these two arithmetical operations are built into the structure of the words used to name this class of numbers.

One significant example of the daily use of these operations is the structure of the Aztec calendar system. The Aztec calendar combined two time-keeping systems. The first was a 260-day religious cycle formed by combinations of the 20 sacred day names with the sacred numbers from 1 to 13. The second calendar cycle was a 365 day solar year cycle. The names for any particular day combined the name-dates for its position in both the sacred year and the solar year.

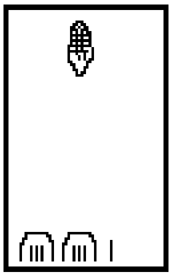
It required 52 solar years for these cycles of the sacred and solar years, known as the Calendar Round, to return to the exact combination of names for the day with which the cycle began. To imagine how this works, think of the Calendar Round as two paired gears, one with 260 teeth and one with 365. Start by placing a dot on each of two adjacent gear teeth (day name components), then start turning the gear pair. It will require 52 turns of the larger gear and 73 turns of the smaller gear before the two dots become adjacent again. Then the cycle of day names that marked the passage of time for the Aztec starts over.

Payne and Closs, along with other knowledgeable scholars, point out that the 52-year cycle, called the xiuhmolpilli [‘sacred bundle’] by the Aztec, was of great significance in Aztec religious life.[50] It formed the basis of Aztec prophesy, often extended over periods of hundreds of years. The conclusion of each such cycle also called for a New Fire Ceremony of purification and renewal throughout the Aztec world.

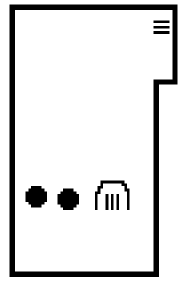
The Texcocan system of notation mentioned in the preceding section was used in post-conquest taxation documents registering land ownership by Texcocan household groups. The system bore a graphic correspondence to the taxable land areas associated with each household and/or adult male.

Harvey and Williams, two scholars who studied the documents, came to understand that the land records used a structural notation that represented farm fields with a somewhat conventionalized rectangular symbol, often with a protuberance in the upper right side. The upper right side constituted the first positional register of the system and showed units (i.e., numbers between 0 and 19). Entries made using lines or bundles of lines on the bottom line of the rectangle are in the first power of 20 (i.e., a single line equals 20, two lines equal 40, a bundle of five equals 100, and so on). The middle of the rectangle was the final register in the second power of 20. Entries were made in this register by use of the dot and the line, so one dot equals 20X20 or 400.

Harvey and Williams found that the second and third registers were never used together. If the third register was empty, the cintli or corn glyph was placed near the top of the middle space in the rectangle. Fractional units (we do not know the size of the fraction) were represented with a glyph for a hand. The area of the field could thus be calculated (roughly, if fractions were present) by adding the value of the first register with that of either the second or third, depending on which was used – big fields required use of the third register, while a field of moderate size could be represented using the second.[51] An illustration showing how this system works appears below.



In this example, the first register is empty (no marks in the upper right side). The second register holds a value of 11X20. The third register (the middle of the rectangle) shows the cintli value of zero. The value of the combined area shown is thus 11x20=220 square units.



Here, the first register shows a value of 3 units. The second register (bottom line) is empty. The third register shows a value of 2x20X20 plus 5X20. The total area is summed 3+800+100=903 square units.

Reasonably precise land measurements and survey computations were of great economic importance for the people of Texcoco, since the Aztec tribute (tax) system of which they were a part was based on land area.[52]

In South America, the Inca of Peru and their contemporaries the Chibcha peoples of Bolivia developed and made use of the famous quipu strings discussed earlier. Quipu strings were organized and stored in state-owned libraries. They appear to have provided a complex, coded system for representing and recording both literary information and the numerical data used in taxing and other administrative functions of the Tawantinsuyu or ‘Inca Empire’.[53]

Varying knot styles and positions, groupings of strings, and the use of color coding enabled Chibcha and Inca specialists to structure the quipu language in a way that may be thought of as analogous to a modern computer language or spreadsheet program.[54] Functionally, the quipus allowed the Inca to create complex numerical records with separate strings functioning like spreadsheet cells and positionally-coded or color coded groups of strings forming linked fields or arrays.[55]

However, the quipu was probably not used directly as a computational device in the way a computer spreadsheet can be used for performing calculations as well as for storing and linking information. Marcia Ascher’s analysis of quipus leads her to believe that while several show summations across an array of positions, the strings do not show evidence of having been used for the actual calculations involved.[56] She concludes that the quipus were used as a data storage system rather than as a computational system.[57]

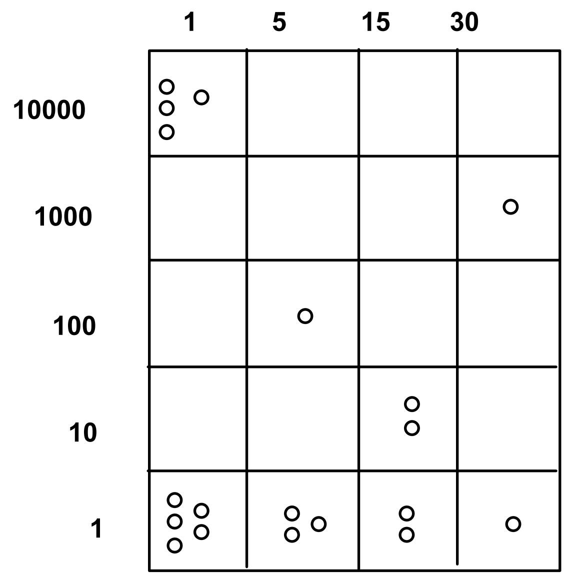
For calculations, the Inca used a counting board, a type of abacus. These boards used scribed lines or internal frames to mark out columns and rows of squares representing place values . Small stones, beans, or corn kernels were employed as markers in this computational scheme, one that was in common commercial and administrative usage.[58]

One of the illustrations in the book **Nueva Crónica y Buen Gobierno** written by the bilingual Inca historian Guamán Poma shows an Inca counting board of this type.[59] Roger Williams of the Connecticut colony reported that 17th century Algonkin villagers also used counting boards in doing arithmetic according to a system of place values; Dr. Gordon Brotherston affirms that counting boards were in use by other cultures from New England to Peru.[60]

The place value system embodied in the counting boards generally varied with the base of the numeration system in any particular culture which used them as a computational device. We know that Inca (a culture with a decimal numeration system) counting boards used ascending powers of ten for the positional value of the rows on their counting boards. The columns were handled differently.

The leftmost column represented units times the power of ten, then to the right came a column representing multiples of five times whatever power of ten was symbolized by the row. Next came a column representing fifteens of that power of ten, and finally on the extreme right of the row was a column representing thirties of the power of ten. The values of these two latter columns on an Inca counting board were determined by divisions of the Inca calendar.

Carrying in Inca counting board computations transformed one counter in the thirties column of any particular row into three counters in the units column one row (or power of ten) higher on the board, according to 17th century Spanish observers.[61] An example of the system is shown here.



In this example, the units row shows the maximum possible counters in each place value column; adding one more counter to a cell in the ones, fives or fifteens columns would require carrying one counter one cell to the right and resetting the original cell to have only one counter. If a second counter were carried into the cell in the thirties column (which can have only zero or one counter), you would carry three counters into the ones column of the next higher power of ten (30 = 3X10). According to the Inca system, then, the units row shows a total value of (5X1 + 3X5 + 2X15 +1X30 =) 80.

What is the total value shown on the counting board in the example?[62]

GEOMETRY

We have only a very limited understanding of the geometry concepts and techniques developed by American Indian cultures in times before Contact with Europeans. Some scholars have provided analyses of American Indian architecture, urban design, and other engineering feats to suggest the kinds of geometrical relations that they recognize in such relics of traditional Indian cultures. Since we have only fragments of the oral and literary tradition that records the original conceptions of these buildings, roads, and city plans, it is hard to know whether we interpret what we can see today in the same way our ancestors did.

Some scholars have probed the surviving oral tradition of various American Indian cultures, seeking an understanding of native geometry. One such scholar is J. Peter Denny, who studied Ojibwe thought about shapes.[63]

Denny concludes from the kinds of terms used by the Ojibwe that members of this culture mentally organize shapes into categories based on selected properties. These properties are reflected in the terms used to refer to the shape categories: the prefix noonim–, for example, conveys the category 'round and elongated'. Denny remarks that "[t]he shapes of the natural world are both irregular and highly variable, so they cannot be efficiently grasped in geometrical terms."[64] (Except, perhaps, in the near approximations available in modern computer-calculated fractal geometries. Even then, efficiency of expression is frequently low when the forms are complex and computation-intensive.)

The Ojibwe express their recognition of this quality of natural forms by naming the variable and complex shapes of their world with terms that have indefinite boundaries and a corresponding variability. According to Denny, Ojibwe geometry terms "show not only the abstractness of the shape categories but their intelligent use in analyzing compound shapes."[65]

Francine Vinette mentions a number of examples of geometry techniques that have been identified or deduced by scholars studying the American Indian cultures of Mesoamerica.[66] Some of these techniques are mentioned in documentary sources while others are deduced from the alignments of structures or the layout of art works. Vinette notes that archaeologists and scholars have identified several varieties of Aztec compasses for laying out arcs and circles.[67] They also possessed and used plumb bobs, levels, builder's squares, trowels and wedges.[68] These tools suggest a fairly sophisticated grasp of applied geometry on the part of Aztec engineers, masons and architects.

Others who have analyzed Mesoamerican artworks such as murals, sculptures and elements of buildings report that they have found markings or relationships between areas of color or in the placement of objects that suggest an intent to divide or place the objects into bilaterally symmetrical arrangements. Vinette and the scholars on whom she relies caution that more seems to be going on than simple geometrical arrangement of space and form in these works of art; they note that aspects of symbolism and religious iconography are also involved. These must be considered as well in any effort to approach the understandings of the native artists, or misunderstanding will likely result.[69]

Vinette mentions several buildings and urban alignments that Anthony Aveni, Horst Hartung and other archeoastronomers have shown to have precise geometric forms, as at Tikal. There, the Maya laid out several temples along baselines from their sides that join them into isosceles right triangles with very close to perfect spacing of the apex points.[70] She offers several other examples and notes that astronomical alignments may not have been the only reasons for Mesoamerican builders to control the placing of structures.

She points out as one example that noted regularities in the orientation and spacing of streets and major structures in the ancient center of Teotihuacan may be evidence of a custom in the urban design and site planning of that metropolis.[71] (Features of Teotihuacan's urban alignments are discussed in the Sciences essay. René Millon extensively mapped this city of over nine square miles and has shown that a sophisticated and ramified type of urban planning went into its design and alignments.[72])

Michael Closs also mentions several other American Indian cultures that demonstrated knowledge and techniques of geometry about which we have some surviving information. The material in the following paragraphs comes from a draft of his 1977 paper "Mathematical Development in the New World."[73]

The Omaha laid out the circular form used in the construction of their earth lodges by driving a stake at what was to be the center point of the new structure. A rawhide cord 10 to 30 feet in length was fastened to the stake with a scribe tied at the other end. Stretching out the cord gave a radius, and walking the scribe around with its point marking the earth marked out a fairly precise circle of 20 to 60 feet diameter.[74]

Similarly, Closs mentions that the Kwakiutl of Vancouver Island used a configuration of pegs and cords to lay out the plan for square houses. As reported by Franz Boas in the 19th century and reinterpreted by a modern scholar,[75] the builders would start by driving two stakes to define a line marking the centers of the front and rear walls of the house. They would then stretch a cord between these two stakes and, having obtained the distance, double the cord on itself to identify its midpoint. With the midpoint known, it is placed at one of the two stakes with the cord's ends extended roughly perpendicular to the line marked out by the two stakes. A second cord is run from the second stake consecutively to each of the first cord's ends to make sure that the ends are located precisely to bring the cord exactly perpendicular to the line between the two stakes. These endpoints are then marked and the whole process repeated with the first cord centered on the second stake to locate the remaining two corner points of the square. [You may wish to give this description to your students and ask them to verify the technique preferably with pegs and cords or by drawing out the procedure to prove that it does yield a square with a reasonably high degree of accuracy!]

While it is unknown how the masons of the pre-Incan Peruvian highland Chavín communities calculated or measured out right angles, evidently they had mastered some technique for squaring the faces of their building stones with great accuracy, according to C.A. Burland.[76]

Some American Indian buildings, especially those which appear to have had observatory functions among their uses, provide suggestive evidence of their cultures' mastery of geometric alignments with astronomical phenomena. Examples appear in the Astronomy section of the Sciences Essay; Closs mentions two worth noting here.

Anthony Aveni, a prominent archeoastronomer, has shown that the 'Governor's Palace' in the Mayan center of Uxmal displays one such alignment. He reports that a line running out from the main doorway and perpendicular to the front face of the building will intersect a small mound 6 kilometers distant and that this will align with the southernmost declination of Venus' rising on the horizon.[77] Thus, the building and the mound are located and aligned geometrically with respect to the celestial manifestation of Kukulcán (Quetzalcóatl), one of the most prominent entities in the Mayan religious pantheon.

Horst Hartung similarly found that a Maya temple at Piedras Negras displays several interrelated geometrical alignments. Its central doorway is oriented due east relative to the center of the adjacent ceremonial ballcourt. The interior 'Altar 2' is oriented due south. From the point of view of the central doorway, the center of the ballcourt and ‘Altar 2’ are equidistant. Thus, the central doorway, the center of the ballcourt and Altar 2 are arranged as the apexes of an isosceles right triangle having lines projecting from two adjacent sides oriented to the four cardinal directions, an alignment with religious significance.[78]

The study of American Indian art, particularly weaving, can offer students a rich opportunity to recognize and identify a number of geometrical transformations. Tessellations, rotations of a figure, symmetries and reflections are some of the characteristic patterns employed by American Indian artists of many cultures in their painting, ceramic and textile arts. Some suggestions for using American Indian figures in studying geometric transformations appeared in an article by Lyn Taylor, Ellen Stevens, John Peregory and Barbara Bath in a recent edition of *Arithmetic Teacher*.[79]

Teachers should be cautious about asserting that Indian artists traditionally thought of these patterns in terms of mathematics, however. We do not know how the figures and effects were originally conceived, but we do know from statments by many living Indian artists that they don’t count out or calculate such effects, they simply know how to do them and use them for esthetic purposes.

Similarly, teachers can use some of the dice and guessing games mentioned in sources in the Physical Education/Health essay to help students grasp some skills in estimating and calculating the probability of possible outcomes. Again, it is advisable to avoid telling students that Indian players traditionally thought in terms of ‘the odds’ when playing such games, as we don’t know that for sure. What we do know is that many modern Indian players rely on intuition and a ‘feel’ for the game more than on calculations or estimates of the probabilities of a given outcome.

CONTEMPORARY AMERICAN INDIAN MATHEMATICS

Information on current American Indian mathematicians is scanty, but suggests a continuing emphasis on applied mathematics (as distinguished from theoretical mathematics) among Indians in the United States. In researching this essay, data were accessible only for American Indian mathematicians and a limited range of applied math professionals in the United States. I am unable to indicate how American Indian participation in contemporary mathematics may differ in other countries or professions.

One recent indication of American Indian participation in mathematics comes from a survey conducted in 1987 of the membership of the National Council of Teachers of Mathematics, a professional association with over 75,000 members. Out of a randomly sampled population numbering 2,200 members, 37 (not quite 1.7%) indicated that they were American Indians. The NCTM includes as members many (but far from all) U.S. K–12 math teachers, two-year college math instructors and mathematics professors in teacher education programs in four-year colleges. If the survey proportion holds for the entire membership, then the NCTM association included approximately 1,260 American Indian members in 1987.

If the membership of the National Council of Teachers of Mathematics represents in its ethnic makeup the population of mathematics teachers in the United States (it may not, being a voluntary association rather than a random sample), the survey results suggest that American Indians are over-represented in the professional ranks of mathematics teachers. The approximate proportion of American Indians in the total U.S. population is three-quarters of one percent, so the Indian proportion of NCTM membership is more than twice as high as might be expected.

There are other indicators of American Indian involvement in some professions that apply mathematics. The American Indian Science and Engineering Society, for instance, had in 1991 a total membership of 2,091 and a professional (practicing engineer or scientist) membership of 220, or about 10.5% of its membership.

U.S. data on American Indian graduates of post-secondary programs in engineering at the associate and bachelor's degree level for 1986–87 showed 332 earning associate-level (A.S.) engineering degrees (about 10% of all American Indian associate degrees earned that year). 214 Indian students earned the Bachelor of Science degree in engineering, or about 5% of all Indian bachelors degrees awarded in 1987. These figures show a strong orientation toward applied mathematics on the part of American Indian students and scientific/technical professionals and suggest the clearly practical orientation of American Indians to contemporary mathematics. They also suggest that well-educated American Indians are more likely than college graduates of most other ethnic groups to have prepared for a career involving applied mathematics.

The principal indicator of American Indian participation in theoretical and applied mathematics at the highest level is the number of Indian people who have earned the Ph.D. in mathematics. So far as is known, the first American Indian to earn the Ph.D. in mathematics in the United States was Dr. Tom Storer, a Navajo who completed his studies in 1964 at the University of Southern California and who currently teaches at the University of Michigan in Ann Arbor. Between 1973 and 1991, a total of 35 other Indians earned their doctoral degrees in mathematics, according to data supplied by the Mathematics Association of America.

APPENDIX A CHRONOLOGY

A Chronology of American Indian Achievements in Mathematics

Dates in **boldface** indicate events primarily due to American Indian initiatives; dates in plain type indicate events primarily due to initiatives by others.

ca. -800 Stonemasons of the Chavín culture in the Peruvian highlands leave evidence in their work that they have mastered a technique for laying out and cutting accurate right angle faces onto their building stones.

ca. -600 At the Zapotec site now called San José Mogote, one of the earliest known American Indian date glyphs is carved onto a threshold stone in a corridor between two public buildings.

ca. -250 At their capitol city now called Monte Albán, the Zapotec carve symbols for the bar-and-dot numerals and the day names of the Calendar Round system which later becomes commonplace in many Mesoamerican cultures.

-36 A stone slab is incised with a date corresponding to December 8 of this year at the late Olmec site of Chiapa de Corzo in what is now the Mexican state of Chiapas.

-32 The Olmec at their center of Tres Zapotes carve a date corresponding to September 3 of this year on a stele.

ca. 0 Maya mathematicians invent a symbol for the concept of zero which they use in their system of positional notation for writing numerals. Their implementation of the concept is more modern than that of the Neo–Babylonians, earliest known users of a symbol for zero. [The Neo–Babylonians are thought to have developed their symbol around 300 B.C., but they used it only in internal positions in numerals, never in the units position. After the Maya, the next culture to independently derive the zero was the Hindus sometime around 800 A.D.]

199 The Maya carve and erect their earliest known dated stele. From this time until the collapse of the classic Maya communities in the late 9th century, dynastic histories and important ceremonial events will be recorded by the Maya on these stelae, which are known from every important Mayan center.

ca. 1100 to 1200 At an undetermined point in this period, the Inca are the probable inventors of the quipu system for recording numbers, numerical labels and other coded information using knotted, color-coded and hierarchically-grouped strings.

1613 The Inca historian Guamán Poma sends his bilingual book **New Chronicle and Good Government** to Spain's King Philip III. In the book, Poma includes an illustration of an object which many mathematicians take to be an Inca counting board, a form of abacus. There are also illustrations showing quipu strings being used and read as accounting records.

1964 Tom Storer, Navajo, becomes the first American Indian known to have earned the Ph.D. in mathematics at the University of Southern California.

1976 Edna Lee Paisano, a Nez Percé/Laguna Pueblo woman, becomes the first full-time American Indian employee of the U.S. Census Bureau. Holder of a master's degree in sociology and trained in statistics, she develops expertise in population demographics and computer programming in the course of her work for the Bureau.

1980 Edna Lee Paisano is responsible for the development of a special questionnaire used in the 1980 census to gather the most extensive data to date on American Indians and Alaska Natives who live on reservations or in the former reservation areas in Oklahoma.

1987 A random sample survey of 2,200 members of the National Council of Teachers of Mathematics reveals that 1.7% (roughly 37 individuals) report themselves as American Indian. The NCTM membership principally consists of elementary and middle school teachers of mathematics. The percentage of American Indian members returned by this sample suggests that American Indian mathematics teachers who belong to NCTM (a highly motivated and self-selected group of professionals) represent about twice the number to be expected from the proportion of American Indians in the general population of the United States.

1990 Edna Lee Paisano, Nez Percé/Laguna Pueblo, heads the division of the U.S. Bureau of the Census responsible for preparing, conducting and analyzing that portion of the 1990 census designed to gather information on American Indian population characteristics.

1991 According to the Mathematics Association of America, 36 American Indians have earned the Ph.D. in mathematics in the United States since Tom Storer (Navajo) first attained the doctoral degree in this field in 1964. Recent Indian doctoral students who have completed their degrees in mathematics include Leonard Huff (Delaware), Margaret Land (Choctaw/Pawnee), Freda Locklear (Lumbee), Claudette Bradley

(Schaghticoke) and Robert Meganson (Oglala Lakota).

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Footnotes

[1]Closs, Michael P. (ed.), **Native American Mathematics.** Austin: University of Texas Press, 1986.
[2]Closs, Michael P., "Native American Number Systems" in Closs, p. 3. This article is hereinafter distinguished from other Closs essays as "NANS".
[3]Closs, Michael P., "NANS", in Closs, p. 42.
[4]Folan, William, "Calendrical and Numerical Systems of the Nootka" in Closs, p. 101-106.
[5]Closs, Michael P., "NANS", in Closs, p. 27-30.
[6]Beeler, Madison S., "Chumash Numerals" in Closs, p. 109-128.
[7]Closs, Michael P., "NANS", in Closs, p. 18-24.
[8]Denny, J. Peter, "Cultural Ecology of Mathematics" in Closs, p. 132-140.
[9]Closs, Michael P., "NANS", in Closs, p. 3-4.
[10]Closs, Michael P., "NANS", in Closs, p. 16-18.
[11]Gnerre, Maurizio C., "Some Notes on Quantification and Numerals in an Amazon Indian Language" in Closs, p. 76-79.
[12]Closs, Michael P., "NANS", in Closs, p.5-10. Closs also provides more than 30 lists of counting terms for a variety of tribes throughout the Americas in his 1977 American Association for the Advancement of Science Conference paper "Mathematical Development in the New World". He drew from 19th and early century linguistic studies by W.C. Eells ("Number Systems of the North American Indians" in *The American Mathematical Monthly*, vol. 20, (1912), p. 263-299); Francis Barnum (**Grammatical Fundamentals of the Inuit Language**. Boston: publisher not given, 1901); j. Hammond Trumbull ("On Numerals in American Indian Languages, and the Indian Mode of Counting" in *American Philological Association, Transactions and Proceedings*, vol. 5 (1874), p. 41-76) and Henry Rowe Schoolcraft **History, Condition and Prospects of the Indian Tribes of the United States**. Philadelphia: publisher not given, in six volumes, 1851-57). Closs also consults a number of more recent linguistic studies; see the references in his volume.
[13]Closs, Michael P., "NANS", in Closs, p. 11-12.
[14]Closs, Michael P., "NANS", in Closs, p. 13-14.
[15]Closs, Michael P., "NANS", in Closs, p. 15.

[16]Murray, William Breen, “Numerical Representations in Rock Art” in Closs, p. 47-54.
[17]Murray, William Breen, in Closs, p. 54-67.
[18]Closs, Michael, “Number in Ojibwe Pictography” in Closs, p. 181-211. This article is hereinafter distinguished from other Closs essays as “NOP”.
[19]Closs, Michael, “NOP”, in Closs, p. 183-185.
[20]Closs, Michael P., “Mathematical Notation of the Maya” in Closs, p. 299. This article is hereinafter distinguished from other Closs essays as “MNM”.
[21]Stodola, Janet, **Mathematical Contributions of the Mayas, Aztecs and Incas**. ERIC Document ED 057 930, May, 1971, p. 9.
[22]Salmoral, Manuel Lucena, **America 1492: Portrait of a Continent 500 Years Ago**. New York: Facts on File, Inc., 1990, p. 162. The earlier Olmec and Zapotec versions of the bar-and-dot notation system lacked a symbol for zero, so far as we know.
[23]Seidenberg, A., “Zero in the Mayan Numerical Notation”, in Closs, p. 374-375.
[24]Torguson, Edgar A., **The Maya Calendar: A Native American Curriculum Unit for Middle and High School**. ERIC Document ED 057 929, May, 1971, p. 8. See also Seidenberg, A., in Closs, p. 376, who gives a documented eighth century A.D. date for the Hindu use of the modern symbol 0.
[25]León-Portilla, Miguel, **Time and Reality in the Thought of the Maya.** Norman: University of Oklahoma Press, 1988, p. 43.
[26]Closs, Michael, “MNM”, in Closs, p. 293.
[27]Closs, Michael, “MNM”, in Closs, p. 294. The spellings given for some of the names of the numbers in this quotation are variants of those shown in the illustration of the bar-and-dot system.
[28]Closs, Michael, “MNM”, in Closs, p. 293.
[29]Gates, William, **An Outline Dictionary of Maya Glyphs**. New York: Dover Publications, Inc., 1931, 1978, p. 86.
[30]Stodola, Janet, p. 9.
[31]Folan, William, in Closs, p. 106-107.
[32]Flornoy, Bertrand, **The World of the Inca.** (Santa Monica: The Vanguard Press, 1965, p. 113), cited in Stodola, Janet, p. 15. Comments in brackets are added by the author for clarification.
[33]The work referred to is Ascher, Marcia and Ascher, Robert, **Code of the Quipu: A Study in Media, Mathematics and Culture.** Ann Arbor: University of Michigan Press, 1981. This expository work draws on an earlier book by the Aschers in which they present the detailed data from their analysis of 190 quipu which they studied in detail. the earlier volume is the **Code of the Quipu Databook**. Ann Arbor: University of Michigan Press, 1978. (Hereinafter cited as Ascher and Ascher.)
[34]See illustrations reproduced in Ascher, Marcia, “Mathematical Ideas of the Incas” in Closs, p. 275-277.
[35]Ascher, Marcia, in Closs, p. 266-268.
[36]Ascher and Ascher, p. 61.
[37]Ascher, Marcia, in Closs, p. 267-269.
[38]Ascher, Marcia, in Closs, p. 270.
[39]Ascher, Marcia, in Closs, p. 271.
[40]Ascher, Marcia, in Closs, p. 273.
[41]Ascher, Marcia, in Closs, p. 271-272.
[42]Ascher, Marcia, in Closs, p. 273-274. Material in brackets is used by the author to clarify the language used in the original.
[43]Ascher and Ascher, p. 80-155.
[44]Brotherston, Gordon, **Image of the New World**. New York: Thames and Hudson, 1979, p. 17.
[45]Ascher, Marcia, in Closs, p. 266.
[46]Harvey, Herbert R. and Williams, Barbara J., “Aztec Numerical Glyphs” in Closs, p. 237-238.
[47]Harvey, Herbert R. and Williams, Barbara J., in Closs, p. 243, 246.
[48]Payne, Stanley E. and Closs, Michael, “Aztec Numbers and Their Uses” in Closs, p. 217. I have corrected the final factor in the Nahuatl expression, since Payne and Closs mistakenly give the term for 18 instead of the Nahuatl term for 17.
[49]Nahuatl speakers use om as an alternate form of the expression on when talking about compound numbers within the first power of 20. The form on appears preferred for talking about combining between

the first and higher powers of twenty. The term for combining between the second and higher powers of twenty, as shown in the example, is ipan.
[50]Payne, Stanley E. and Closs, Michael, in Closs, p. 222-224.
[51]Harvey, Herbert R. and Williams, Barbara J., in Closs, p. 244-246.
[52]Harvey, Herbert R. and Williams, Barbara J., in Closs, p. 249-254.
[53]Brundage, Burr, **Lords of Cuzco**. Norman: University of Oklahoma Press, 1967, 1985, p. 116.
[54]Ascher, Marcia, in Closs, p. 282-283.
[55]Olien, Michael, **Latin Americans: Contemporary Peoples and Their Cultural Traditions.** New York: Holt, Rinehart and Winston, Inc., 1973, p. 53.
[56]Ascher, Marcia, in Closs, p. 278-282.
[57]Ascher, Marcia, in Closs, p. 281.
[58]Olien, p. 53, quoting Inca historian Garcilaso de la Vega.
[59]Ascher and Ascher, p. 63.
[60]Brotherston, Gordon, p. 126.
[61]Brotherston, Gordon, citing the Jesuit Father Joseph Acosta in Samuel Purchas' five volumes of reports from immigrants to the Western Hemisphere in 1625-26, p. 127.
[62]70,880. Starting from the units row we have the following: 80 + 300 + 500 + 30000 + 40000.
[63]Denny, J. Peter, in Closs, p. 162-168.
[64]Denny, J. Peter, in Closs, p. 164.
[65]Denny, J. Peter, in Closs, p. 165.
[66]Vinette, Francine, "In Search of Mesoamerican Geometry" in Closs, p. 387-407.
[67]Vinette, Francine, in Closs, p. 388.
[68]Vinette, Francine, in Closs, p. 388-389.
[69]Vinette, Francine, in Closs, p. 389.
[70]Vinette, Francine, in Closs, p. 394-397.
[71]Vinette, Francine, in Closs, p. 403.
[72]Coe, Michael, p. 90, 92, 100.
[73]Closs, Michael P., "Mathematical Development in the New World". A partial draft of this symposium presentation on Ethnoscience in Native America at a 1977 meeting of the American Association for the Advancement of Science was provided to the author by James Swanson, a high school mathematics teacher at Jefferson High School in the Portland District. It is distinguished from other Closs references below as "MDNW."
[74]Closs, "MDNW", section 7, citing Seidenberg (1962: 522).
[75]Closs, "MDNW", section 7, citing Seidenberg (1962: 521-522).
[76]Closs, "MDNW", section 7, citing Burland (1976: 21).
[77]Closs, "MDNW", section 7, citing Aveni (1975: 184-186).
[78]Closs, "MDNW", section 7, citing Hartung (1975: 193).
[79]Taylor, Lyn, Stevens, Ellen, Peregory, John and Bath, Barbara, "American Indians, Mathematical Attitudes and the Standards". *Arithmetic Teacher*, Vol. 38, February 1991, p. 14-21.

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