Asian Contributions to Mathematics

By

Ramesh Gangolli

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AUTHOR: Gangolli SUBJECT: Math

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1. Introduction.

1.1 Purpose of this essay.

This essay, like others in this series of baseline essays, has educators at the levels K-12 as its intended audience. Its structure and scope are influenced by my conception of the nature and purpose of this series of baseline essays; namely, that the series seeks to provide educators at levels K-12 with a perspective that will increase their awareness of the contributions made by different cultures to human knowledge in general and to U.S. society in particular. The aim is to enable them to incorporate in their teaching a point of view that is appreciative of all cultures and their contributions. One can hope that as a result of being informed by such a point of view, their students will grow to appreciate and respect different cultural elements in our society, and will be able to deal more effectively with an environment that is rapidly becoming culturally more diverse. The importance of this for our society cannot be overstated.

In conformity with the conception described above, I have set for myself the following two aims:

A. To provide a perspective on contributions to the discipline of mathematics by people whose cultural heritage is Asian. I shall try to provide this perspective by looking at the history of mathematics and examining some of the junctures at which Asian cultures have played a significant role by contributing ideas and techniques to the subject.

B. To provide educators with a sketch of the role played by Asian-American mathematicians in present day mathematics in the United States. A brief perspective on the development of U.S. mathematics will be included in the course of this sketch.

1.2 Scope.

In addressing the first aim described in the preceding section, I am very much aware of the vast reach of the subject, and of my own limitations with respect to it. In selecting the contributions and topics to be discussed, I have been guided by considerations such as: Does the contribution represent a conceptual or technical landmark? Does the topic serve the audience for which this essay is written; e.g., does it make a connection with an aspect of the K-12 curriculum? Does it tell us something about the development or transmission of mathematical ideas? Nevertheless, the material presented can only be a thumbnail sketch of the total material available.

In addressing the second aim, again I cannot be anywhere nearly as complete as some might wish. The development of science and mathematics in the United States in the last century has been astonishing and has brought the United States to a pre-eminent position in the world in scientific and mathematical research. It would be impossible (and unnecessary) to describe the specific disciplinary content of this development in an essay such as this. But, I have taken the view that educators (and their students) will benefit from a brief narrative of the general nature of the development of U.S. mathematics, and especially the role played in it by diverse groups of immigrants, including Asian-Americans.

In describing the contributions made by Asian cultures, another limitation arises from the vastness of Asian history and the great variety of peoples and subcultures that constitute Asia. It is obvious that it would be impossible to talk about "Asian" contributions (no matter how "Asian" is defined) except in terms of grossly aggregated cultural/geographic categories. I have restricted myself to three major Asian geographical areas in which (at different times) a significant mathematical tradition was built up by local peoples whose linguistic, cultural and intellectual ethos is acknowledged as Asian. These three culture-areas (to borrow Needham's phrase) are:

China, India, and the area in West Asia that consists of present day Iraq, Iran, and segments of neighboring regions. This last area I shall loosely refer to as West Asia, for convenience. I should mention here that I have found it difficult to resolve a real problem of terminology with reference to this area. Many scholars have often referred to the empire of the Caliphs of the Umayyad and ‘Abbasid dynasties as the Arab empire and have referred to its culture as Arab culture. Although convenient, it is widely realized that this is problematic. Although the Caliphs themselves were Arabs, the scholars at their court (especially during the ‘Abbasid period) represented a wide mix of backgrounds and geographic origins, from Arabia, Greece, Persia, Turkey, Syria, Central Asia, and Northern Africa. Thus, the culture of the court was a mixture of many influences, and an attempt to label it by a short name is bound to cause a problem. Given this situation, I have chosen to refer to this culture-area as West Asia, sometimes qualifying it with the historical period mentioned above, and referring to scholars of this period as West Asian scholars of the Umayyad and/or ‘Abbasid periods. I have not tried to identify the geographical origin of each individual scholar whose name is cited.[1] These disclaimers should be borne in mind by the reader.

I realize that the omission of other culture-areas of Asia is unfortunate. This is partly because enough scholarly attention has not been given to the study of the history of mathematics in those regions, leading to a lack of sources of information. But the major reason for the omission is, of course, my feeling that I totally lack the competence to deal with the history of mathematics in those regions.

1.3 Acknowledgments.

I have benefited from the comments of the following persons (listed in alphabetical order of last names) who were kind enough to read through drafts of the essay and to make suggestions for its improvement: Dr. Mariam Baradar, Joseph Chang, Dr. Benny Cheng, Dr. Tracy Dillon, Aseel Nasir Dyck, Professor S. Nomanul

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1 Similarly, I have generally guarded against the temptation of referring to the culture of the West Asian empire of this epoch as Islamic culture. I feel that this is not appropriate because the scientific ethos from which the scholarly contributions came cannot be said to have a religious basis. (Indeed, many of the scholars were not even Moslem, but merely subjects of an Islamic regime.) However, many scholars have found it convenient to refer to the empire of the Caliphs as the Islamic empire, or the Arab/Islamic empire.

Haq, Professor Shirley Hune, Professor Elaheh Kheirandish, Chris Landon, Carolyn M. Leonard, Professor Li Di, Kanta Luthra, Luis Machorro, Michael Meo, Susan Montag, Vinh Nguyen, Professor Nosratollah Rassekh, Professor Frank Swetz, and Minh Tran. I would like to thank them for the care and concern with which they made their suggestions. I have not been able to incorporate all their comments in this essay, for a variety of reasons. However, I have incorporated most of their comments and I hope that the essay as it now stands will serve its purpose adequately. Any deficiencies that remain are, of course, mine alone.

2. Some views of the development of Mathematics.

Mathematics is central to the intellectual history of all societies. Its origins go so far back into prehistory that it is impossible to pinpoint the epochs in which the first inquiries in it (such as formation of the concept of number) took place. Its factual content is probably as culture free as any product of human activity can be. By this I mean that the discovery, discussion and evaluation of mathematical facts has an objective basis independent of the conditioning of any particular culture.[2] In this respect, mathematics differs from many fields of human activity, in which cultural factors often present insuperable barriers to communication. This circumstance enables mathematics to have a universal appeal, so that its study is often pursued as an enterprise shared by a community that is truly global. This is perhaps not surprising in the context in which we live today, when communication is so rapid and relatively easy. But even in times past, when such was not the case, one can see examples of this sense of a community that transcends geographical and cultural boundaries.

Although the subject matter of mathematics has an objective basis in the sense described above, our view of the development of mathematics is inevitably influenced by our own predispositions and prejudices, which are often the result of our educational and cultural background. Consider, for example, the following view about the development of mathematics, contained either explicitly or implicitly in many texts.[3] For the sake of reference I will label it as the orthodox view.

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2 I do not intend this as a formal philosophical statement. In particular, I am aware that cultural factors, especially language, indeed have an influence on the kind of inquiry on which mathematicians might choose to focus, and that certain mathematical ideas accepted at a given time are influenced by the cultural milieu in which they arise, only to be modifed radically later. However, in the short run there is a body of mathematical assertions whose meaning and interpretation are understood independently of the culture of the interpreter. Thus, a mathematical assertion such as "two plus two equals four" will have the same meaning or validity (allowing for differences in language) in any culture. This is also true of many other mathematical assertions. Of course, individuals will often bring their own particular biases to their evaluation, but these reside largely with the individual, and not with the subject.

3 For a detailed treatment of this topic I refer the reader to Joseph (1993).

The orthodox view

- The beginnings of deductive mathematics in antiquity were in Greece. Greek investigators (mainly in the period from about 600 B.C. to 300 A.D.) gave a logical basis to the subject.
- After the "Greek miracle," there was a period of stagnation, the so-called dark ages, of a thousand years or more.
- At the end of the dark ages, there was a rediscovery of Greek learning that led to the Renaissance in the 15th and the 16th centuries. The foundations of modern mathematics were laid during the Renaissance.
- The post-Renaissance development of mathematics, from the 17th century onwards, took place essentially in Europe.

Here are some observations about this view. (a) It is focused entirely on Europe and does not take into account the contributions of other cultures (such as the ancient river valley civilizations of West Asia, Egypt, China, India, the medieval cultures including Persia, Turkey, and various parts of the Arab empire, and pre-Columbian America) to the development of mathematics. (b) As a result, the first three assertions given above are over-simplistic and ignore the continuous cultural diffusion that has been taking place between Asia and Europe over the centuries. (c) The last assertion is substantially correct. However, the extraordinary achievements of European mathematicians in the 16th through the 19th centuries and the relative lack of such achievements in other parts of the world during that period need to be put in the perspective of the political and economic history of the world over the last four hundred years. Ideas such as the role played by the emergence of colonialism and the Industrial revolution and their effects on non-European societies inevitably become relevant in this context.

I think that one can safely say that the orthodox view held sway in most European nations and also in the United States until about the middle of this century.[4] The realization that this view was incomplete, and in particular that it dealt summarily

4 A typical example is the treatment in Dickson (1919-23), a standard work of its time. It was Struik (1987) who first showed a more eclectic viewpoint.

with the contributions of other cultures in antiquity and in medieval times, came in the forties and fifties. This was in large measure due to the discoveries made in the late 19th century and the early 20th century by European and U.S. scholars about the river valley cultures of antiquity (Sumeria, Babylon, Egypt and so on). The orthodox view was slowly replaced by a slightly more sophisticated one, which I shall call the neo-orthodox view.

The neo-orthodox view

- The beginnings of mathematics in antiquity were in Greece. Greek investigators (mainly in the period from about 600 B.C. to 300 A.D.) gave a logical basis to the subject. However, Greek mathematics incorporated knowledge from cultures going back several centuries before the advent of Greek science — for example, Babylon and Egypt. The precise extent of interaction between these cultures and Greece is not known, but it is clear that these cultures had a recognizable mathematical tradition, and that it had an effect on Greek thought.
- After the "Greek miracle," there was a period of stagnation in Europe, the so-called dark ages, of a thousand years or more. During the period from the 7th through the 13th centuries A.D., a culture of active scholarship in many fields of endeavor, and in particular in mathematics, developed in West Asia during the reign of the Umayyad and 'Abbasid Caliphs. These scholars, drawn from many parts of the pan-Islamic world,[5] studied and translated Greek works. Thus, they acted as repositories of Greek knowledge during the dark ages in Europe. During the 11th and 12th centuries, they transmitted this knowledge back to Europe, together with some discoveries that they themselves had made.
- At the end of the dark ages, there was a rediscovery of Greek learning in Europe that led to the Renaissance in the 15th and the 16th centuries. The interaction between Europe and West Asia through Spain helped this process in many ways. The foundations of modern mathematics were laid during the Renaissance.
- The post-Renaissance development of mathematics, from the 17th to the 19th centuries, took place essentially in Europe.

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5 One should bear in mind that scholars of the Umayyad and 'Abbasid courts came from a vast geographical expanse, including Persia, Turkey, and several parts of central Asia, as well as parts of Northern Africa.

This view, while acknowledging the existence and priority of other traditions besides the Graeco-Roman, still accords primacy to the latter and oversimplifies the possible interactions between various traditions that have played a significant role in the development of mathematics. It continues to be a view molded by the implicit assumption that the history of science has, essentially, a European locus. I believe that it is probably substantially the view that our educational system transmits at present; in other words, it is probably close to the view that is held by a majority of persons in our society who have some interest and knowledge about this circle of questions.

In the last five or six decades, the work done by a number of scholars (R.C. Gupta, Otto Neugebauer, Joseph Needham, A. Seidenberg, B.L. van der Waerden, and D.B. Wagner, to mention just a few) in the history of science in China, India and in West Asia has led to a much better understanding of the level of mathematical sophistication that many non-European cultures had attained, and also of the complex interactions among different cultures that probably took place in antiquity and through the middle ages. This has led to what one may call the modern view. See for example, Needham (1959), Neugebauer (1975), Joseph (1983), Pingree (1973, 1978, 1989) and the references given there. One can summarize the modern view as follows.

The modern view

- In antiquity, one finds several distinct streams in the development of mathematics, stemming from many different cultures: Babylon, Egypt, Persia, China, India and Greece; one can infer that certain mathematical ideas diffused from some of these cultures to later cultures. But we do not know, and perhaps may never know, the precise mechanism and extent of the transmission.
- Greek ideas, especially in geometry and on the nature of numbers, were decisive for the development of mathematical thought. They formed a starting point for many later independent investigations by mathematicians in other cultures, especially in India and in West Asia during the period of the Caliphs. The mathematicians of the latter period (drawn from all parts of the Islamic world) not only preserved Greek knowledge but added to it in many areas, especially in algebra. They also assimilated many ideas from India and China, where mathematical knowledge had developed.

- Questions of priority in mathematical ideas are often troublesome, and our views at any given time are dependent on the then current state of the historical record. The view will change as historical research yields more information.[6] In several cases where the same mathematical idea seems to have been investigated in two different cultures widely separated spatially, and when the historical record does not indicate plausible transmission from one to the other, it is reasonable to conclude that the discoveries were probably made independently by scholars in different cultures.[7]
- We need to continue to investigate the intellectual exchange between different cultures in forming our ideas about the history of science and mathematics, and we must realize that our view of that history at any particular time is based on the state of our archaeological and historical knowledge at that time.[8]

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6 Indeed, historical research in the past hundred years has uncovered parallel (not necessarily contemporaneous) streams of investigation in different cultures. Many basic facts (for example, what we refer to as the theorem of Pythagoras concerning the lengths of the hypotenuse and the sides of a right triangle, techniques of solving certain simple algebraic equations, algorithms for fast multiplication or division, fundamental combinatorial facts which are referred to today as Pascal's triangle, and so on) which were considered at the beginning of this century as purely European in origin are now known to have been the subject of investigations by mathematicians in China, India, or Western Asia, at different phases in history. In several cases, these investigations predate the period in which European mathematics considered the same issues. It is reasonable to conclude that some of the discoveries were probably made independently by scholars in different cultures.

7 To be sure, this is a view that is not universally held. See, for example, van der Waerden (1983), where it is argued that all major mathematical discoveries are probably made once and only once. According to this view, the occurrence of the same significant mathematical fact in two different cultures is evidence of a common ancestral source from which that fact permeated into these cultures. In the case of two far-flung cultures whose connections and commonalities in other fields of thought are very tenuous, it seems to me difficult to accept this hypothesis without further proof. I prefer to think that intelligent people in several cultures were probably able to think along similar lines and were led to the same basic facts.

8 New discoveries can quickly and radically modify our ideas, however long held. Witness the effect of Leakey's discoveries at Olduvai gorge, concerning early *Homo sapiens,* on the long held view that the first examples of *Homo sapiens* were from the Cro-Magnon group.

3. The place of mathematics in ancient cultures.

Every culture, no matter how rudimentary its daily life is, needs to resort to counting in some way. Counting is such a fundamental human activity that almost all cultures of which we have knowledge have had to use some system of enumeration. The concept of number and the process of counting is buried deep in our past. It is probably futile to expect to get a detailed understanding of the origin and the development of concept of number from the historical record. But it is clear that even in groups in which the needs were basic and rudimentary, there must have been many uses for the process of counting: whether it be for counting friends or enemies, or children, or items of food (e.g., game). The most simple of systems of enumeration may have very few conceptual underpinnings: for example, we know that there were certain tribes in Papua-New Guinea who used only the rudimentary concept of one-many as the basis of their numeration system.[9] Yet others had only the concepts of one, two and many as the basis of their system. In such cultures, there would have been no need to make any detailed comparisons of size or magnitude. Many other cultures, which were basically nomadic and whose primary methods of survival were centered on food gathering or a combination of hunting and food gathering, also had numeration systems that were simple, incorporating just the level of complexity that was demanded by the structure of their lives.[10]

As societies became more complex, the management of the greater variety of activities undertaken necessitated more sophisticated tools for enumeration and for measurement. Accompanying the need for developing these tools of enumeration and measurement, there was also the need to devise effective and easily remembered methods of representing and communicating the results of such measurement. The emergence of agriculture and the subsequent transition from a nomadic to an agrarian society seem to have been crucial in providing this impetus. The agrarian society, rooted to a stable location, was able to produce a wider range of goods (for consumption as well as for trade). The natural annual cycle of an agrarian society afforded periods of leisure during seasons when agricultural work was slack, thus

9 See, for example Eves (1983). Another reference is McLeish (1991).
10 For an excellent survey of this topic, the reader is referred to Conant (1923).

providing opportunities for a variety of activities that nomadic societies could not sustain. In turn, these activities provided the extra impetus needed to stimulate the development of more versatile methods of calculation and representation than were available in the previous nomadic phase.

One can easily see that as an agrarian society develops in complexity, it experiences many needs that would call for better methods of calculation and measurement. What we know of early cultures enables us to identify three strands of motivation that have had an impact on the development of mathematics in those societies. These are: the need for mathematics in agriculture, astronomy, commerce, architecture and engineering; the use of mathematics as an esoteric science in connection with ritual (whether in a religious context or otherwise); and finally, the pursuit of mathematics purely for its own sake, as a recreation as well as a branch of knowledge. I discuss these briefly here.

3.1 Agriculture, commerce, etc.

Among the activities that are associated with these fields, one can list several that would require the development of some sort of mathematics for their orderly execution.

- Managing the timing of activities such as sowing, tending, reaping, etc., relating to the crop.
- Gauging the size of the crop and managing its proper allocation for consumption and (possibly) trade.
- Designing and building facilities for the storage of crops.
- Determining reasonable terms of trade for barter or sale.
- Developing means of transport for the purpose of trade; this would be a very powerful impetus, and would, by extension, include activities such as roadbuilding, shipbuilding, navigation and cartography and so on.
- Deciding issues of inheritance of property, especially land.

This is not, of course, an exhaustive list, and is included just to clarify the connection between the natural course of development of economic activity in an agrarian society and the stimulus for the development of mathematics that it provides.

Agriculture, commerce, architecture and engineering are the themes that unite these activities. It is not hard to see the connection between these activities and various types of mathematics that would be useful in supporting them. One only needs to consider the use of numeral systems and arithmetic in commerce, of geometry in architecture, and of trigonometry in surveying, navigation and cartography to appreciate the connection immediately.

The earliest civilizations that we know about arose out of agrarian societies that developed in the great river plains and valleys of the temperate climatic belts on our planet. The Nile valley in Egypt, the valleys of the Tigris and Euphrates in West Asia, the Indus valley and the plains of the Ganges and Jamuna in India, the valleys of the Yangtse and the Yellow River in China, as well as the valley of Mexico and the coastal valleys of Peru, were all cradles of early agrarian societies that wrestled with problems of agriculture, trade and architecture/engineering arising in a natural manner out of their economic development. These societies also tried to deal with the problems of social and political organization that emerged when social and political units such as cities developed in these societies. It is not surprising that what we know of the early history of mathematics is tied to our knowledge of these cultures, and in particular encompasses the types of mathematical subjects that developed in these cultures.

3.2 Ritual, worship, etc.

In addition to the needs of agriculture and commerce, another factor probably had a significant influence on the development of human societies in the post-agrarian ages. This is the concern with ritual, and especially its presumed role in the processes of divination and prediction. Occult crafts such as divination and augury are, roughly speaking, rooted in the same soil as science, namely in the desire of human beings to make sense of the world around them and their place in it, and to control that world. In most early societies one finds evidence of ritual practices aimed at influencing observable phenomena that had a significant impact on their lives, but of which they had no clear understanding. The regulated poetry of the sun, moon and the stars; eclipses, comets and other "strange" events; the cycle of the seasons; the mysteries of fertility and growth; the larger imponderables of birth and death; these are some examples of the types of phenomena that early societies sought to understand and control.[11]

The models that early cultures built to understand these phenomena posited that these phenomena had power to influence human lives and postulated supernatural forces that governed them. Not surprisingly, this led to magical prescriptions for the propitiation of these forces so that their detrimental influence on human life could be averted or minimized. Sometimes, the seemingly magical properties of numbers and geometrical figures seemed like appropriate tools by which these forces could be propitiated. Thus, erection of pyramids, henges or other large structures that had a ritual connection with certain astronomical events such as solstices; constructions of altars to magically specified proportions or in certain geometrical shapes; painstakingly accurate drawing of certain magical patterns; prescriptions to use certain objects or incantations in sets of seven (or three, or some other number regarded as auspicious or magical) — these were some of the ways in which mathematical considerations impinged on these aspects of people's lives and led sometimes to investigations of numbers or geometrical figures in the context of their applications to ritual needs.[12]

3.3 Intellectual curiosity and amusement.

The creative impulse in mathematics, as in many other types of human intellectual activity, often arises from playful inquiry, coming from no motive other than that of satisfying one's sense of curiosity and amusing oneself. Undoubtedly, the stimuli provided by the two types of forces mentioned above (namely economic and religious) were extremely important in the development of mathematics in ancient societies. But one should also be aware that this third source of inspiration, based on human curiosity and playfulness, has played an equally significant role in the development of mathematics, and indeed of all the sciences. Properties of numbers and sets of numbers have a fascination that we all feel as children or adults. Mathematical riddles or puzzles have had a place in the miscellany of every culture. Games of chance have oiled (or thrown a stick in) the wheels of the social machinery of

11 Even as we try to do today! Indeed, this whole paragraph might apply with only slightly altered force to most present day societies as well.
12 See Seidenberg (1963).

every culture. Many mathematical discoveries have been motivated by the desire to understand games and puzzles. Even when the initial impetus to study a particular mathematical problem comes from a particular "real world" problem, one soon finds that the mathematical tools that one needs cannot be focused merely on the particular problem from which they arose. Thus, along with properties of a mathematical concept or object that eventually get used in the context of the application that inspired the problem, one discovers many other properties that are intrinsic to that concept or object. These additional discoveries may not be of immediate use in the problem at hand but can become a part of the conceptual infrastructure of the subject, awaiting a future time at which they might become useful. This serendipitous aspect of mathematics, and of all basic science, has been observed repeatedly in the intellectual history of human societies, and it will be seen below how it has played a role in the development of mathematics as well.

4. Asian contributions in specific mathematical areas.

4.1 Counting, number systems and calculation.

4.1.1 Counting.

Early societies that did not have complex needs nevertheless resorted to some rudimentary form of counting. The very structure of languages, in which the singular and plural are essentially always differentiated, gives us evidence of the awareness and formalization of the number concept via vocal sound. Indeed, there are many examples of languages in which the constructed forms of nouns, verbs, etc. are governed by three numbers: the singular, the dual and the plural. These devices are, of course, vocal indications of the notions of one-many or one-two-many, and may be loosely called vocal counting devices. (Our language even today contains examples of purely vocal ways of indicating the notion of the number two, without specifically referring to the number two. For example, we refer to two horses as a *team* of horses, to two partridges as a *brace* of partridges, to two weeks as a *fortnight*, etc.)

Although the most rudimentary societies probably got by with no more than these primitive concepts of number, and perhaps used only vocal counting, one can imagine that simple tally methods, such as making marks on a surface or notches in a piece of wood corresponding to the set of objects being counted (e.g., two marks or notches for two sheep, etc.), probably emerged very early in the history of human societies. The plausibility of this line of development, and of its evolution into some sort of symbolic representation of number, is to some extent supported by the work of anthropologists in our own time, who have studied contemporary societies that are at a rudimentary stage of development.

4.1.2 Number systems.

By the time the economic organization of society got even a little more complex, the need for better counting methodology must have been felt quite keenly. The earliest systematizations of the counting process use a notion substantially similar to what would today be called a base (also called a radix). A number, say r, was selected as a base, and names were chosen for the numbers represented today by the symbols 1, 2,..., *r*. The numbers that are larger than *r* were then represented by some combinations of the symbols already chosen. The decimal system that is in common use today is an example of the use of the base ten. The choice of ten as a base is fairly natural, since human beings have ten fingers in all. (Five is also a fairly natural choice, and indeed, the ancient Chinese system, as exemplified by the abacus, used five as a base.) Examples of other bases used by different societies are quite easy to obtain. Eves (1983, p. 4) cites several examples of the use of other bases including 2, 3, 4, 5, 12, and 20. As is well known, 20 was the base used by the Mayan cultures in their calculations of astronomical events. Sixty was the base used by the Babylonians in their system of enumeration.

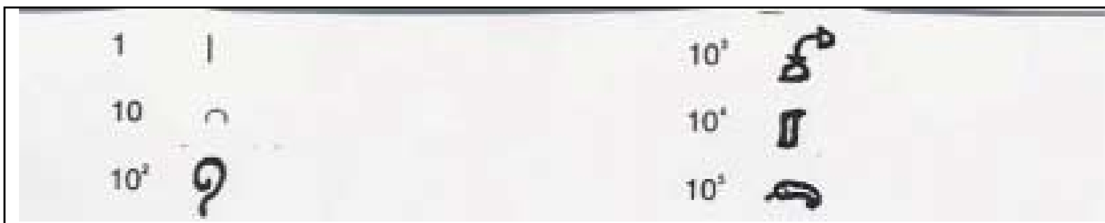
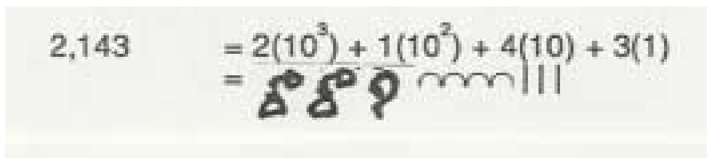


Figure 1
Facsimile of Hieroglyphic numerals as displayed in the Rhind Papyrus
Based on Chace (1927)

The earliest systems for the symbolic representation of numbers were developed in the great river-valley civilizations of Egypt and of West Asia. The earliest Egyptian system dates to about 3000 B.C., and is what is called a simple grouping system. Its base is 10. In this system, a symbol is chosen for each power of ten — i.e., for 1, 10, 10^2, 10^3, 10^4, etc. Any number is expressed by combining these symbols, repeated as often as needed to represent that number. For example, in Figure 1, there are distinct symbols for 1, 10, 10^2 and 10^3.

The number that is represented by 2,143 in our present day system would be expressed in terms of these symbols as follows:



Similarly, a grouping system based on the numbers 10 and 60 was used by the early Babylonians, who used characters knows as cuneiform (meaning wedge-shaped) characters.[13] The writing medium was clay, and a stylus that made a depression in the soft clay was used as a writing instrument. The stylus made an impression in the shape of a triangle, and the shape and the orientation of the triangular impression could be controlled by tilting the stylus in different ways. Thus, symbols standing for different numbers could be made from these triangles, of different shape and orientation. These symbols in turn served as the basic building blocks of a simple grouping system of number representation (Figure 2a).

In Figure 2a, one has a combination of place value and grouping. The groups are 1 (the first symbol); 24 (the next 6 symbols); 22 (the next 4 symbols); and 21 (the last 3 symbols). Their placement indicates powers of 60.[14]

After imprinting the clay tablet with impressions of this type, the tablet was baked to a hard durable state. Tablets of this kind from the period 2000 B.C. to 200 B.C. have

13 The sexagesimal system (based on 60) was in use as long ago as 2000 B.C. in Babylonia. This knowledge passed directly into Greek knowledge, and certain conventional practices of measurement (even in our day) can be traced directly to this strand of transmission. As examples we may consider the division of the hour into 60 minutes, the minute into 60 seconds, the circle into 360 degrees, and the year into 360 days (in earlier versions of the calendar).

14 Thus, the leftmost group represents 1 times the third power of 60, the next group to the right represents 24 times the second power of 60, and so on, as spelled out above.

been recovered by archaeologists and give us an idea of the progress made in the realm of calculation by these early cultures.

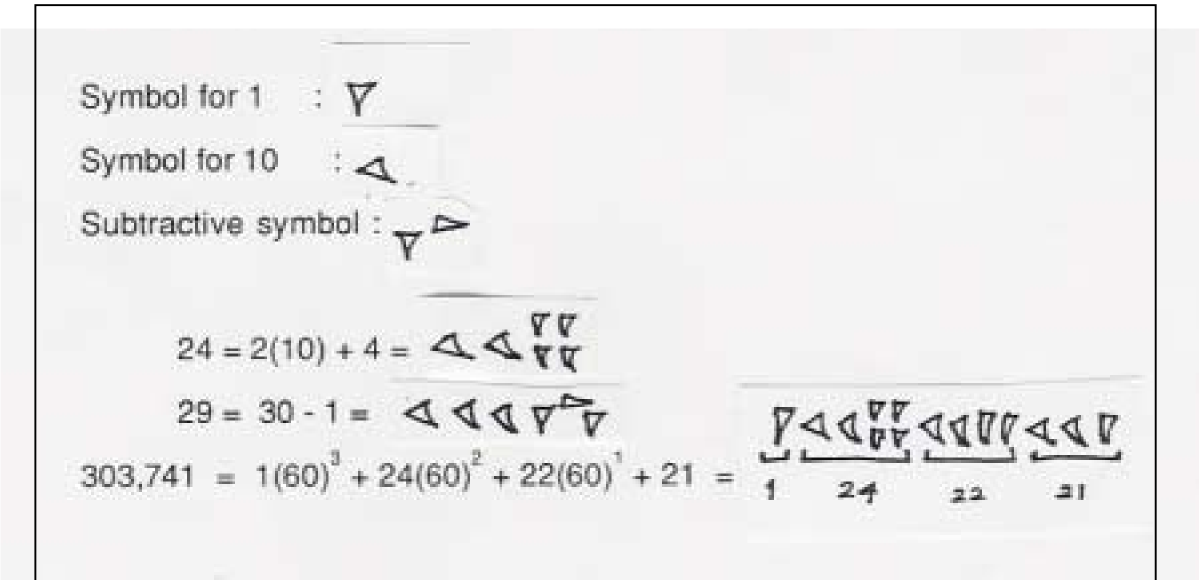


Figure 2a: Cuneiform numerals

Simple grouping systems such as the ones described above were probably adequate for daily commerce, but a little reflection convinces one that they are basically unwieldy for the representation of large numbers, and cumbersome for calculation. As social organization became more complex, other systems of number representation that might make calculation more convenient were tried in different cultures. Notably, in China, an enumeration system was developed several hundred years before the advent of the Christian era. It contained the idea that is at the core of the place value system that is in general use today. The numerals that were used in this system were called rod forms or rod numerals, and are illustrated in Figures 2b and 2c. This system developed further between the 2nd and 4th centuries A.D. and was described in the *Sun Tzu Suan Ching* (*Master Sun's Arithmetical Manual*, ca. 3rd century A.D.).[15]

15 See also Lam and Se (1992) for more information.

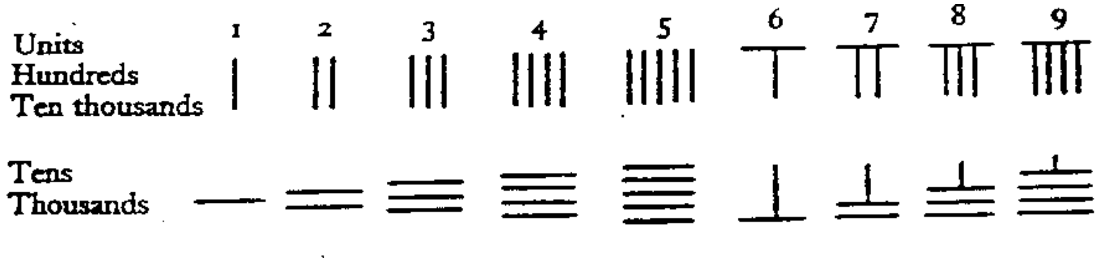


Figure 2b: Rod numerals
As shown in Needham (1959, p. 9)

Originally a mixture of a simple grouping system together with a concept of place value, it evolved through a series of stages into a fully positional system that was eventually adopted as the basis of enumeration by several neighbors of China, such as Japan, Korea, and Viet Nam. (The various stages of development of this system have been called enumeration systems in their own right.) By the 13th century, a fully positional base 10 system using a symbol for zero was in general use in China. These enumeration systems using rod numerals formed the base for most computational methods in China until about 1600 A.D.

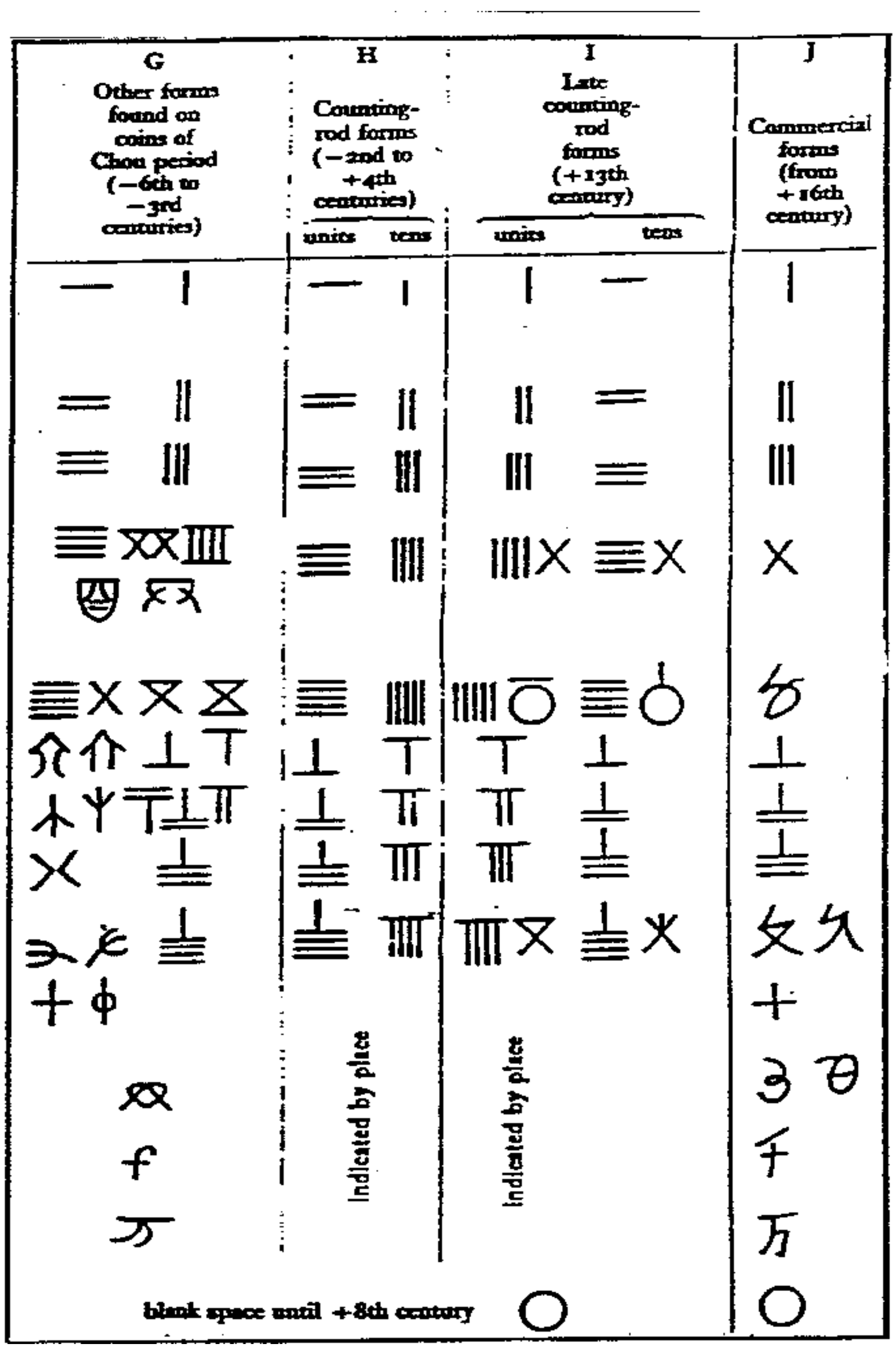


Figure 2c
Rod numerals from China
As shown in Needham (1959, p. 8)

Indeed, such numerals are used in the traditional Chinese system of number representation even in our time. In the 13th century A.D., the Chinese mathematician Li Yeh also mentions a notation for the concept of negative number.[16] Although these systems were quite well developed for the purposes of everyday commerce, they still lacked a symbol for zero and had not developed a full place value system such as is in use today.

Figure 2d illustrates the Roman numeral system.

Symbol for 1	: I	Symbol for 100	: C		
Symbol for 5	:	V	Symbol for 500	:	D
Symbol for 10	:	X	Symbol for 1000	:	M
Symbol for 50	:	L			

Figure 2d
Roman numerals and their use

In early examples of the use of Roman numerals, a number was represented by simple grouping. For example, 1996 would be represented as:

1996 = 1000 + 500 + 400 + 50 + 40 + 5 + 1 = MDCCCCLXXXXVI

In the last four or five centuries, a subtractive principle came into common use, and this principle is still in use today. It is hard to say exactly when this practice started. This subtractive principle represents 900 as 1000 - 100, and the subtraction is denoted by putting the symbol for 100 *before* the symbol for 1000. Thus 900 = CM. Similarly, 90 would be represented as 100 - 10, putting the symbol for 10 before the symbol for 100;

16 However, the concept of negative number was known long before this to Chinese mathematicians. For example, it occurs already in the 2nd century classic *Nine Chapters* where issues of representation of negative numbers, as well as calculation involving them, are discussed.

thus, one would write 90 = XC. In this practice, which is current practice for Roman numerals, we have

1996 = 1000 + 900 + 90 + 5 + 1 = MCMXCVI

Arithmetic operations with Roman numerals are quite cumbersome, and therefore prone to error.

A landmark in the development of number representation systems is surely the development of the Hindu-Arabic system of numerals, which is the system that is in use at the present time. (See Figure 2e.)

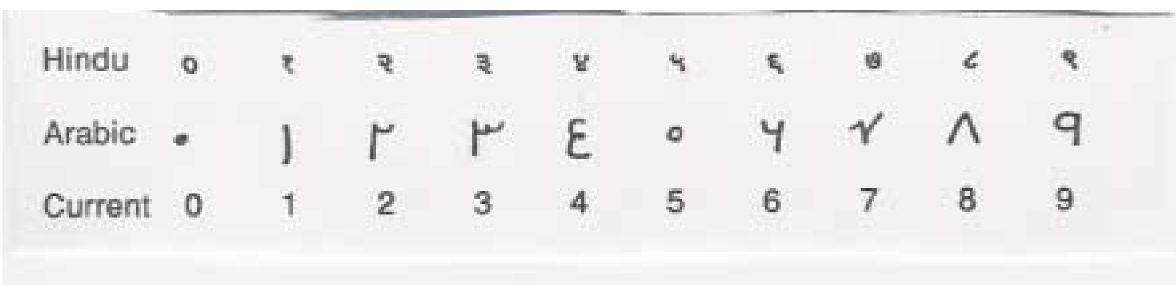


Figure 2e

Hindu-Arabic decimal numerals[17]

The origin of this system cannot be accurately pinpointed. The complete Hindu-Arabic system, as it stands today, probably emerged between 400 A.D. and 700 A.D. in India, and was transmitted to scholars in West Asia during the ‘Abbasid period, a transmission no doubt facilitated by the extensive trade between West Asia and India for several centuries. In turn, these scholars acted as transmitters of this system to the west. The earliest known archaeological example of the use of the Hindu-Arabic numerals in their present form is found on some stone columns dating from around 250

17 Hindu decimal numerals are in the *Devanagari* script, as used in Sanskrit. Arabic decimal numerals are in the Arabic script.

B.C., erected by the Indian emperor Asoka.[18] However, the use of the numerals made in these early examples does not yet indicate an understanding of the notion of place value, nor is there any use of the concept of zero. (See Eves, 1983, p.14; Conant, 1923.)

The distinguishing features of this system are: (a) it is a place value system — i.e., the value of a symbol is highly dependent on the position in which it is placed. (b) The system has a symbol (zero) for the concept of "nothing" and is a base 10 system. We all know how to use this system. Thus, the symbols 1, 2, 3, ... 9 serve as the basic digits representing numbers one through nine, augmented by the symbol 0, the zero, for the concept of "nothing" or "void". Any number N can be expressed by means of a string of digits, $a_n\ a_{n-1}\ ...\ a_1$, a_2 , a_0, where each symbol a_1 , a_2 , a_3 , etc. is a digit between 0 and 9 (except the leading digit a_n, which cannot be 0) and the string of digits

$$a_n\ a_{n-1}\ ...\ a_2\ a_1\ a_0$$

stands for the number

$$a_n\,(10^n) + a_{n-1}\,(10^{n-1}) + ... + a_2\,(10^2) + a_1\,(10^1) + a_0\,(1)$$

Of course, the success of this method of number representation depends on the fact that every whole number can be expressed in one and only one way as a sum of multiples of powers of ten. The various multiples that occur in such an expression form the digits in the representation of that number in digits.[19]

For example, the string of digits 523 stands for

$$5(10^2) + 2(10^1) + 3(1)$$

---

18 The ancient Chinese system was also a place value system. However, in its early versions, there seems to have been no symbol for zero. A blank space was used for the zero instead. By the tenth century, the Chinese system had started using the same symbol for the zero as the Hindu-Arabic system uses. During the Sung dynasty, the Chinese system also began to use a specific device to denote negative numbers, by putting a slanting slash on the numeral in the unit's place. I owe this comment to Professor Li Di, one of the reviewers of the first draft of this essay. See also Needham (1959), and Mikami (1913).

19 Another system was developed by the Mayans. The system combines features of simple grouping system with a notion of place value, and also has a symbol for zero. Details of the Mayan system can be found in Macleish (1991) and Thompson (1941). See also Aveni (1980).

Thus, the *position* of the digit 5 determines the *value* that it contributes to the number being represented, i.e., the five is in the "hundreds" place, so that it contributes $500 = 5(10^2)$; the position of the digit 2 determines that its contribution is $20 = 2(10^1)$ — i.e., that the 2 is in the "tens" place, etc. Notice that the string of digits 253 has a completely different meaning from 523, exactly because the place occupied by the digits 5 and 2 in the two strings are different. This is the essence of the concept of place value, and it is a concept that makes a startling difference in the ease with which computations could be performed. It also liberated one's imagination from the limitations of notation in the sense that large numbers could now be not only conceptualized but also notated and handled on essentially the same footing as small numbers. This was in marked contrast with earlier systems in which large numbers could only be notated in a cumbersome way. It is easy to underestimate the significance of this subtle change in perception. It made possible for people to tame astronomical magnitudes and converse about them with a facility that was hitherto denied to them due to purely technical reasons of limitation of symbolic notation.

From its emergence in India, the system migrated to West Asia by the end of the ninth century. The Persian mathematician al-Khwarizmi explains it in his treatise dated 825 A.D. From West Asia, the system slowly diffused into Europe. While one does not at present know precisely the details of this process of diffusion, one can conjecture that it came into Spain through the expansion of the Arab empire westward (as evidenced by the use of the Hindu-Arabic system in some 10th century manuscripts in Spain), and thence spread into Europe. The treatise of al-Khwarizmi was translated by Gerard of Cremona from Arabic into Latin in the 12th century A.D., served as a textbook in many European universities until the 16th century A.D., and thus was largely the medium through which the dissemination of the Hindu-Arabic numeral system into Europe took place. Although medieval Arabic manuscripts referred to this system as the "Hindi" system, it came to be known in Europe as the Arabic system of numerals. By the end of the 15th century, it had replaced the system of Roman numerals used in Europe.

4.1.3 Calculation.

The processes of addition, subtraction, multiplication and division are fundamental to arithmetic, and are of great utility in everyday life as well as in science. Much thought must have gone into the devising of systems or representation and of procedures that would make arithmetic tractable. Our knowledge of older cultures shows that there were two principal modes of performing calculations in the past. In some cultures, perhaps partially because of the scarcity of writing materials, or because of the lack of a good number representation system, used concrete objects such as beads, rods, etc. to facilitate the keeping of tallies. The prime example of this approach is the use of the Abacus. (see Figure 3.)

Other cultures developed towards symbolic representation by means of some form of writing, experimenting with a variety of media: bone or tortoise shell (China), papyrus (Egypt), clay tablets (Babylon), bamboo slips (China), parchment or leather (Medieval Europe, West Asia), palm leaves or cloth (India). Paper made from linen and rags was invented in China very early (beginning of the 2nd century A.D.). The technique of paper production diffused to other parts of the world from China, and paper was used as a writing medium in many parts of the world. But it was very expensive due to the nature of the raw materials used in its production, which were themselves end products of a complicated manufacturing process. (Paper as we know it today, rolled from wood pulp, is an infant in the family of writing media, having come on the scene only a little over a century ago).

Using these radically different approaches — the first tactile and concrete, the second symbolic and abstract — algorithms (i.e., step-by-step procedures) for addition, subtraction, multiplication and division were devised in different cultures. Each algorithm had to be adapted to the peculiarities of the counting system on which it relied. The algorithms devised for the abacus in medieval China are still in use today in the marketplaces of Asia and Africa. In the other direction, algorithms for arithmetic operations, using various symbolic representation methods, starting from the Egyptian and Babylonian systems, and ranging through the Greek and Roman numeral systems to the Hindu-Arabic system, are well documented in archaeological sources. Work done by many historians of mathematics gives us a clear and detailed picture of these procedures. In the confines of this essay, I cannot give details of the various procedures that were used in antiquity. However, I should mention that the procedures paid careful attention to concepts such as carrying or borrowing in addition and subtraction, shifting due to placements of digits in multiplication, and the carrying of digits in long division.[20]

In Figure 3, the lower illustration shows the number 5,857 represented on the abacus. A bead touching the bamboo (going across the abacus) but above the bamboo represents 5. A bead touching the bamboo but below it represents 1. Thus, in the lower figure, the four verticals respectively represent 5, 8, 5 and 7.

The abacus was in general use in West Asia by the 8th century A.D., and procedures for its use were widely disseminated there. The long division algorithm that is today called the Euclidean algorithm was also (and probably independently) known to the Chinese, as evidenced in the famous Chinese book *Chiu Chang Suan Shu (Nine Chapters on the Mathematical Art)* dating to the period 200 B.C.-200 A.D.[21] Algorithms for early Chinese versions of the abacus were probably in general use in China by about the 5th century A.D.[22] Procedures for calculation with rod numerals

20 See Joseph (1993) and Eves (1983, p. 162 ff.) for a more detailed discussion of these matters.

21 The author of this work is not known. It is cited as Anonymous in the bibliography. A German translation does exist. (see Vogel, 1968.) This work will be referred to as *Nine Chapters.*

22 See Needham (1959, p.79) for the chronology of the abacus. He is of the opinion that the Chinese abacus probably predates the European version but notes that one cannot be certain. As to its transmission, he agrees with Sarton's view that they were probably independent inventions.

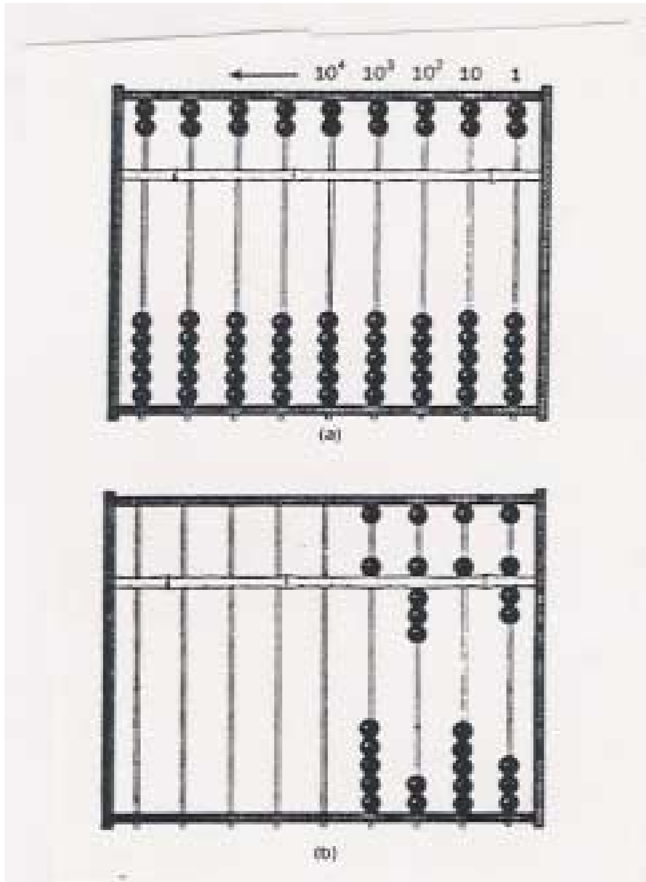


Figure 3
A Chinese abacus
From Joseph (1993, p. 90)

can be found in books by the Chinese mathematicians Chin Kiu-Shao (13th century A.D.) and Li Yeh (13th century A.D.). Sophisticated procedures, essentially equivalent to the way we do multiplication and division today, are to be found in the work of the Indian mathematicians Brahmagupta (7th century A.D.) and Bhaskara II (12th century A.D.). Also found in the 14th century work of Chu Shi-Kie is a diagram that clearly indicates that the elementary properties of what are called binomial coefficients at present (as explicated by "Pascal's triangle") were known in China at that time. Indeed,

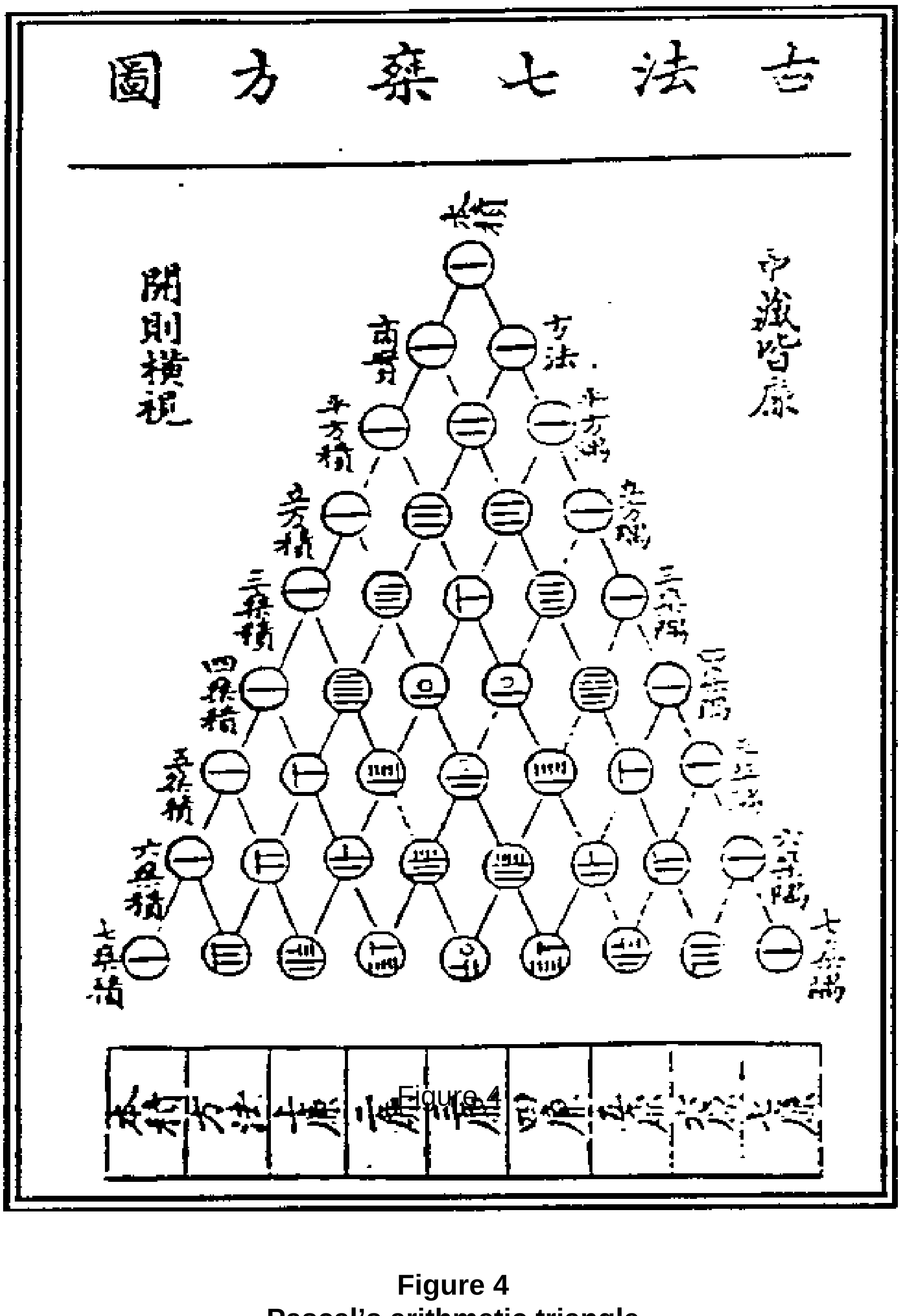


Figure 4
Pascal's arithmetic triangle
From Ssu Yuan Chien (Precious Mirror of Four Elements)
By Chu Shi-Kie (1303 A.D.) As shown in Needham (1959, p. 135)

Chu Shi-Kie was quoting the work of an earlier Chinese mathematician, Jia Xian (11th century), who seems to have also known about the binomial coefficients. (see Figure 4.)[23]

Apart from the elementary arithmetic operations of addition, subtraction, multiplication and division, the mathematicians of the ancient world also investigated more complicated operations such as calculation of square roots. To circumvent the fact that a numerical calculation would be cumbersome in the absence of a good system of representing numbers, the problem of calculating square roots was posed and solved in geometrical terms by the Indian mathematician Baudhayana (ca. 700 B.C.), who was one of the authors of an ancient book named the *Sulvasutras*. Also, in the *Nine Chapters,* there is a perfectly general method of calculating the square root or the cube root of any integer. Moreover, the method given there is remarkable because it enables one to calculate the square root or cube root to as many decimal places as necessary. Of course, the results were not expressed in terms of the decimal system notation of today, but the principles were identical.[24]

The following quotation from van der Waerden (1983, p.46) referring to the Chinese method of extracting square roots as given in the *Nine Chapters* is very interesting: "Whoever found the method to extract square roots and cube roots must have been an excellent mathematician. Moreover, he must have had a decimal number system at his disposal, for his method yields one decimal after the other. In the sexagesimal system the calculations would have been much more complicated."

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23 As is often the case in such matters, the *concepts* very often precede the surviving historical record of the *procedure*. Thus, for example, the rod numerals were known and used in China long before the period of the works (known to us) describing their use. Professor Li Di informs me that in the *Sun Tsu Suan Ching* (Master Sun's 3rd century classic), a description of the use of rod numerals occurs, but without their explicit symbolic representation. See also Needham (1959, pp. 6-11).

24 See van der Waerden (1983, pp. 45-47) for a detailed description of the exact procedures used by these Chinese and Indian works.

4.2 Algebra.

The first book on algebra known to us is the book by the Persian mathematician Muhammad ibn Musa al-Khwarizmi (9th century A.D.). It survives to us today only in Latin versions translated from the Arabic original. The title of the original book in Arabic was *Hisab al-jabr wa-al-muqabala*. The word algebra in fact is derived from the word *al-jabr* in that title. *Hisab* means calculation. By *al-jabr,* which has been translated as "completion or restoration," al-Khwarizmi meant the process in which the same term is added to both sides of an equation in order to eliminate negative terms. He used *al-muqabala*, which has been translated as "reduction" or "balancing," to denote the process in which like terms on both sides of an equation may be canceled. To illustrate, I use the words of van der Waerden, (p. 70):

> Thus, the equation
>
> $$50 + x^2 = 29 + 10x$$
>
> which occurs in Rosen's translation of al-Khwarizmi's Algebra on page 40, is reduced by *al-muqabala* to
>
> $$21 + x^2 = 10x$$
>
> which in Rosen reads: "There remains twenty-one and a square, equal to ten things." [25]

Thus, al-Khwarizmi was concerned in his book with the art of solving equations by "completion, restoration, reduction and balancing." The process by which one can systematically do this came to be called algebra. The subject that is called algebra today has expanded to include other types of problems, but the solution of equations containing one or more unknown quantities remains a basic concern. The fundamentals of algebra were systematically laid out by the Persian mathematicians who were at the court of the Caliphs, between the 9th through the 12th centuries A.D. During these years, the empire of the Caliphs was in the ascendant phase and provided patronage to many scholars, among whom were a number of brilliant mathematicians and astronomers from many parts of the world, especially from Persia. Their contributions and discussions, many of which have survived to our time via Latin translations made

25 The reference here is to Rosen (1831).

later by European scholars, played a fundamental role in shaping the progress of science in Europe during and after the Renaissance.

The work of al-Khwarizmi and other Persian algebraists attempted to describe general procedures of solving classes of equations rather than to provide *ad hoc* recipes for solving a specific puzzle-like problem. This is a departure from the style and intent of earlier investigations which consider questions whose answers hinge on the solution of certain types of equations. For example, word puzzles which are equivalent to solving a set of simultaneous linear equations in two or three unknowns occur in the *Nine Chapters* in China and also in the works of two Indian mathematicians, Aryabhata (ca. 500 A.D.) and Brahmagupta (ca. 630 A.D.). The following problems, the first one from the *Nine Chapters*, the second from Brahmagupta, and the third from an imprecisely dated Indian manuscript (between the 3rd to the 12th century A.D.), are typical:

> The yield of 3 sheaves of superior grain, 2 sheaves of medium grain, and 1 sheaf of inferior grain is 39 *tou*. The yield of 2 sheaves of superior grain, 3 sheaves of medium grain, and 1 sheaf of inferior grain is 34 *tou*. The yield of 1 sheaf of superior grain, 2 sheaves of medium grain, and 3 sheaves of inferior grain is 26 *tou*. What is the yield of superior, medium, and inferior grain?[26]

> Two ascetics lived at the top of a cliff of height 100 *yojanas*, whose base was at a distance of 200 *yojanas* from a neighboring village. One descended the cliff and walked to the village. The other, being a wizard, first flew up to a certain height above the cliff, and from that height flew in a straight line to the village. The total distance traversed by each was the same. Find the height to which the second ascetic ascended.[27]

> A merchant pays duty at three different places on the good that he carries. At the first place he gives 1/3 of the good, at the second 1/4 of the remainder, at the third 1/5 of the remainder. He pays a total duty of 24 pieces. What is the original quantity of the good?[28]

The first problem leads to three linear equations in three unknowns. The third problem leads to a single linear equation, in one unknown, and the second one leads to

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26 This is the translation by D.B. Wagner, as quoted in van der Waerden (1983, p. 47).
27 See Colebrooke (1973, p. 79).
28 This problem is quoted from an undated Indian manuscript known as the *Bakhshali Manuscript*. See Midonick (1965, pp. 92-105).

an equation which at first appears to be quadratic, but turns out to be linear after reduction.[29]

Chinese and Indian mathematicians also considered quadratic equations in one variable. The 20th problem in the *Nine Chapters* is translated by Professor Li Di (written communication) as follows:

> There is a square city of which the size is unknown. At the middle of each side there is a gate. There is a tree at a distance of 20 *bu* from the northern gate. Leave the southern gate for 14 *bu* and turn to the west for 1775 *bu*, then you can see the tree. What is the size of the city?

It can be seen that if x denotes the length of a side of the square city, then x can be found by solving the equation:

$$x^2 + 34x = 40 \times 1775$$ [30]

Similar problems leading to quadratic equations are posed in the work of Indian mathematicians, notably Mahavira (ca. 850 A.D.), Bhaskara II. Indeed, Bhaskara II in his famous book *Lilavati* gives a procedure for the solution of quadratic equations that is substantially equivalent to the present day "quadratic formula." Here are two examples from *Lilavati:* [31]

> A bamboo pole 18 hands high was broken by the wind. Its top touched the ground at a point 6 hands from the root. Tell the lengths of the two segments of the bamboo.[32]

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29 The first problem leads to the following three equations: $3x + 2y + z = 39$; $2x + 3y + z = 34$; $x + 2y + 3z = 26$. Here x, y, z stand respectively for the yield of superior, medium and inferior grain. The third problem leads to the following single linear equation: $x/3 + (x - x/3)/4 + (x - x/3 - x/6)/5 = 24$. Here x stands for the original quantity of the good. The third problem leads to the equation $(100 + x)^2 + 200^2 = (300 - x)^2$, where x stands for the height ascended. This appears to be quadratic equation but reduces to a linear equation when the parentheses are cleared.

30 The Chinese mathematicians Li Yeh (ca. 1248) and Chhin Chiu-Shiao (ca. 1247) also studied certain equations of degree higher than the second.

31 The first one appears to lead to a quadratic equation but can be reduced to a linear equation. The second problem leads to a quadratic equation. In the first problem, let x denote the height at which the bamboo gets broken. Then one sees easily that x satisfies the equation $6^2 + x^2 = (18-x)^2$. This reduces to a linear equation when the parentheses are cleared. In the second problem the problem translates to the equation $x - (x/4 + \sqrt{x}) = 15$, where x stands for the size of the herd. Upon clearing the square root, we get the quadratic equation $4x = [3(20 - x)/4]^2$.

32 Quite possibly, Bhaskara got this problem from China. It was known in China as the Problem of the Broken Bamboo. It occurs in the *Nine Chapters*. The problem was explicated by the 13th century

> One fourth of a herd of camels was seen in the forest; twice the square root of that herd had gone to the mountain slopes, and 3 times 5 camels remained on the riverbank. What is the numerical measure of the herd of camels?

These examples make it clear that linear and quadratic equations were considered by early Chinese and Indian mathematicians.[33] However, in the work of the Persian algebraists, one finds a far more systematic and general approach, which seeks to understand *not only these specific* types of equations, but the *fundamental structure* of algebraic procedures and reasoning. Thus, al-Khwarizmi's book contains systematic procedures and attempts an exposition, in the manner of a treatise, of what he calls the art of calculation. Today, any such systematic procedure is called an algorithm, a word which is a Latin corruption of al-Khwarizmi's name. This change of viewpoint, from an *ad hoc* approach to a systematic one, is one of the chief features of the work of the mathematicians of West Asia.

A significant step in the theory of equations was taken by the Persian mathematician and poet Omar Khayyam (ca. 1100 A.D.) when he considered and solved a cubic equation in his treatise. (see al-Khayyami, 1851.)[34] Although his method of solution was in essence graphical and geometric, and therefore did not advance algebraic techniques *per se*, it opened the door to the possibility of an

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mathematician Yang Hui in a famous picture. (see Figure 5.) Yang Hui's picture has come to us through a 14th century Chinese manuscript of Chu Shi-Kie.

33 In the *Nine Chapters*, one also finds a problem that leads to an underdetermined system of linear equations. Indeed, one finds there a problem involving six unknowns but only five equations. Professor Li Di informs me that the work of Zhang Qiu-Jian entitled "The hundred pheasant problem" also contains a problem leading to two linear equations in three unknowns.

34 Often referred to also as ‘Umar al-Khayyam or Omar al-Khayyam. Note that "al-Khayyami" is now archaic. He is known to the English speaking world today chiefly for the collection of poetic quatrains known as the Rubaiyat. The seventh century Chinese mathematician Wang Xiao Tong in a work entitled "Collection of Ancient Arithmetic" also considered and solved several special cubic equations by using geometric methods. However he does not seem to have arrived at a systematic method for treating all cubic equations graphically. (I owe this comment to Professor Li Di.)

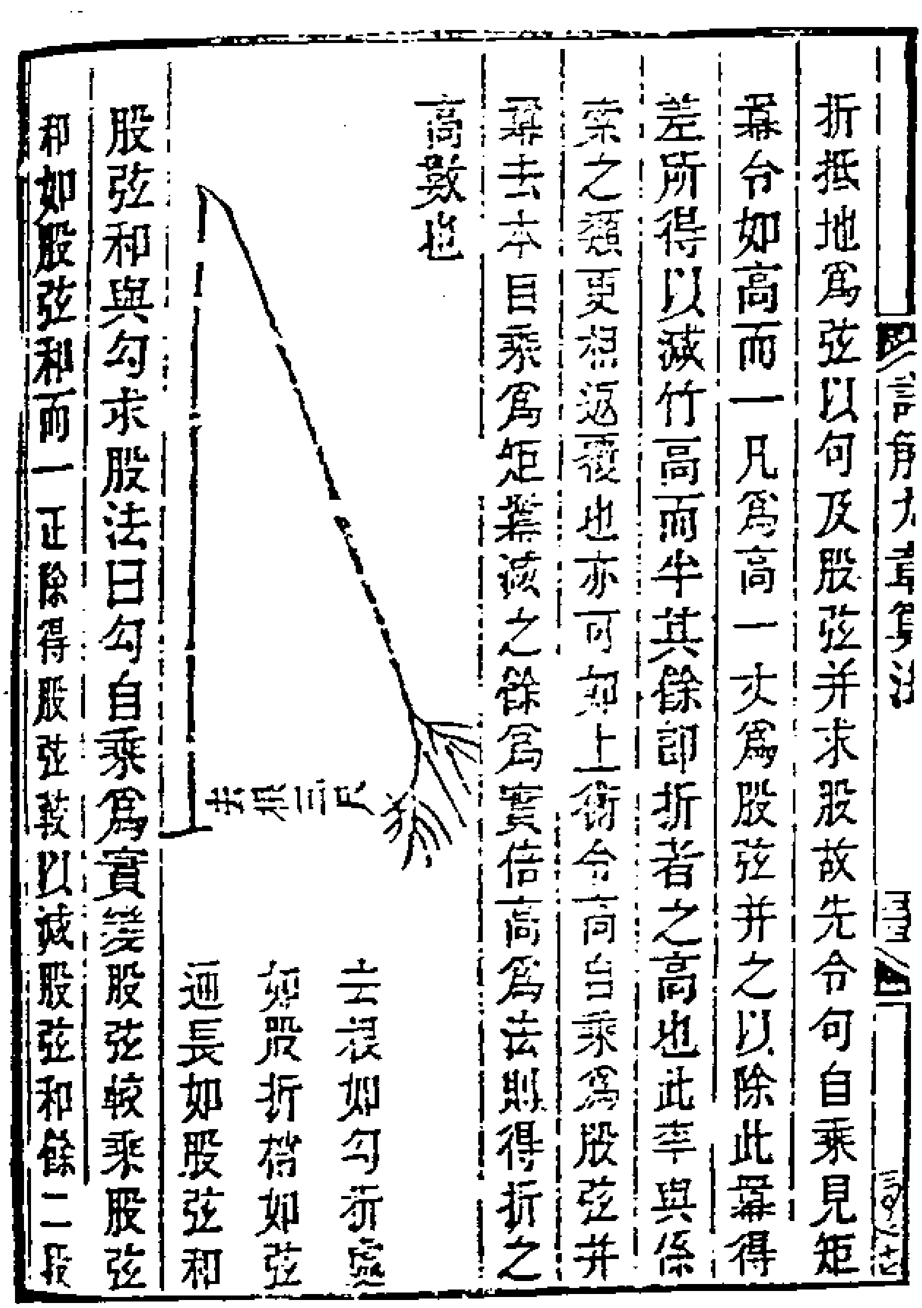

Figure 5
The problem of the broken bamboo
From Hsiang Chieh Chiu Chang Suan Fa, by Yang Hui (1261 A.D.)
As shown in Needham (1959, p. 28)

algebraic approach to the problem of solving cubic equations. Such an approach was provided and the general cubic equation was solved by the Italian mathematicians Tartaglia and Cardano in the 16th century.

4.3 Geometry and trigonometry; calculation of lengths, areas and volumes.

Human beings have been fascinated by the study of geometrical objects and facts concerning them from the earliest times. In early societies, the symmetry and perfection of regular geometric shapes and objects was often associated with mystic power, and maintaining that perfection sometimes formed an integral part of some ritual or the other. References to the magic power of pyramidal shapes occur in ancient Egypt, for example. Also, early Egyptian priests devoted a considerable amount of thought to the problem of constructing regular polygons, for ritual purposes, and had devised a method for constructing several such polygons by ingeniously using knotted cord. In early India, sacrificial altars were supposed to be constructed in certain geometric shapes. Adherence to the particular proportions of that shape was supposed to endow the altar with magical properties that would militate towards the success of the sacrifice. A departure from the shape and its proportions would presumably ensure failure of the ritual. Procedures for the construction of such altars are contained in ancient Indian texts (ca. 650 B.C.) called the *Sulvasutras.*[35] A similar preoccupation is evident in ancient Greece. The three famous mathematical problems of ancient Greece (Duplication of the Cube, Squaring of the Circle, and Trisection of the Angle) were supposed to have come from the Oracle of Delphi and are all geometric in essence. The ascription of these to the Oracle indicates a ritual connection. These are just a few instances of the inspiration provided for the study of geometry by ritual and worship.

Commerce and navigation also provided a powerful impetus. Measuring areas and accurately marking plots of land became important activities as soon as notions such as individual ownership and taxation became a part of social organization. Telling time and accurately planning routes of travel over sea were equally urgent needs, which would lead naturally to geometric questions of diverse sorts.

35 See Seidenberg (1978) for details on the problems addressed therein.

One of the earliest discoveries that we now automatically associate with geometry was the discovery of what are called Pythagorean triples of integers. Three integers (say *a, b, c*) are said to form a Pythagorean triple if the sum of the squares of two of them equals the square of the third integer. For example (3, 4, 5) is a Pythagorean triple because $3^2 + 4^2 = 5^2$. Another Pythagorean triple is (5, 12, 13). Such triples are called Pythagorean because of the geometric fact known today as the theorem of Pythagoras, which states that the sum of the squares of the lengths of the two smaller sides of a right triangle equals the square of the length of the hypotenuse. Although such triples are now named after Pythagoras, the fact that such triples exist in abundance (in fact, one can show that there exist an infinite number of such triples) was known to most early cultures that had any sort of a mathematical tradition, dating back to a time long before Pythagoras. An early Babylonian text (known as Plimpton 322) lists several such triples, and early Greek and Hindu mathematical problems dealing with altar constructions were solved using tools which suggest that the users had a knowledge of these triples, and also of their connection with right triangles.[36] The methods of construction of megalithic monuments in Brittany, Scotland and Ireland, dating back to the period 3000 B.C. to 2000 B.C., show an understanding of Pythagorean triples.[37] In the *Nine Chapters,* one also finds a number of problems dealing with Pythagorean triples. In all cases, not just one or two, but *several* Pythagorean triples are mentioned. Given the fact that such triples are not easy to discover empirically, more or less by accident, one can only conclude that these cultures were not only aware of the properties defining these triples, but were also cognizant of a procedure for generating them. (see, for example, Swetz and Kao, 1977.)

The explicit connection between whole numbers that form Pythagorean triples — i.e., integer triples (*a, b, c*) such that $a^2 + b^2 = c^2$ — and a right triangle whose sides have lengths *a, b, c* seems to have come after the discovery and generation of Pythagorean triples by purely arithmetic means. Certainly Pythagoras knew this connection with geometry. One also finds an explicit and clear mention of it in ancient India in the *Sulvasutras* of Baudhayana (ca. 500 B.C.), which possibly predate

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36 See van der Waerden (1983, Introduction, p. xi). There the author expounds his thesis that these three cultures probably were derived from an older neolithic culture that influenced the mathematical sciences of these cultures.

37 See Wood (1978) cited in van der Waerden (1983). In pp. 16-21 of the latter, one can find an excellent account of the evidence for this view.

Pythagoras. (See van der Waerden, 1983, p.9, for a fuller discussion.) Whatever the origin of the discovery of the theorem of Pythagoras (namely, the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides), one finds that many later Chinese and Indian works devoted considerable effort to different geometrical proofs of that theorem. The standard proof found in most U.S. text books is ascribed to Euclid. However, other pictorial dissection proofs can be found, for instance in the Chinese work on astronomy titled *Chou Pei Suan Ching*, probably of the Han period (200 B.C. - 200 A.D.) or earlier,[38] and in the work of Bhaskara II in India (ca. 12th century A.D.).

The fascination with the theorem of Pythagoras (which continues to this day) is at bottom due to the fact that it connects, in a surprising way, arithmetic and geometry. Ancient mathematicians pursued the direction suggested by this connection and often used geometric arguments and constructions to solve certain algebraic or computational problems. An example is the construction of the so-called mean proportion between two lengths a and b . That is, one wants to find a length x such that

$$a/x = x/b$$

This is equivalent to finding x such that

$$x^2 = ab$$

This problem was reformulated as follows: given a rectangle of sides a and b, find a square whose area equals the area of the rectangle. Once found, the side of such a square will equal the number x sought as the solution to the above equation. This formulation occurs in Greek geometry, in Aristotle, as well as in Indian texts (*Sulvasutras, Satapatha Brahmana*) of the 6th or 7th century B.C.[39] In the latter, the following problem is treated: given a falcon shaped altar of area equal to seven and a half *purusas,* construct an altar of like shape having an area of eight and half *purusas*.

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38 Cited in Needham (1959) with an accompanying picture from the Chinese source. Liu Hui (3rd century A.D.) gave a method of systematically finding *all* integer triples that can be taken as the sides of a right triangle. According to Professor Li Di, Zhao Shuang (3rd century A.D.) also gave a proof of the theorem of Pythagoras.

39 Seidenberg (1963). This is so early that it was probably not influenced by Greek geometry.

In the process of solving this problem, it turns out that it is necessary to solve the problem of constructing a square whose area equals the area of a given rectangle. The problem is then solved in these Indian texts by an ingenious use of the theorem of Pythagoras. The significant idea here is the utilization of geometric concepts and constructions in the pursuit of arithmetic or algebraic problems and vice versa. This idea has proved to be fecund and fundamental in mathematics, and continues to shape the latest developments in mathematics.[40] Other examples of this kind will be given below.

Distinct from the strands stemming from the connection between geometry and algebra/arithmetic, exemplified by the Pythagorean theorem, there was another very important and pervasive theme that ancient studies in geometry pursued. This was based on the remarkable connection between the circumference of a circle and its diameter. We know that the ratio of the circumference of a circle to its diameter is *the same for all circles*. This ratio is an irrational number, denoted by the Greek letter p (pi). The ratio of the area of a circle to the square of its radius also equals the same number, namely π. Again, this ratio is independent of the size of the circle. A great deal is known today about π, but perhaps most people do not realize that the study of this ratio fascinated the best mathematicians of antiquity. Moreover, that study often led to rich dividends, due to the intrinsic significance of the ratio.

The fact that the ratio of the circumference of a circle to its diameter is independent of the size of the circle was known in antiquity. Even the fact that the ratio is a number close to 3 (its actual value, correct to four decimal places, is 3.1416) was clearly understood in ancient times. In the Old Testament there is a reference to a circular altar nine cubits around and three cubits across. In Babylon, China and India, and perhaps also in Egypt, the constancy of the ratio was understood and regarded as a significant mathematical truth, and various attempts were made to calculate the value of π accurately. It is a tribute to the genius of those mathematicians that they arrived at approximations to π which are good even by today's exacting standards. For example, in the Rhind papyrus, an ancient Egyptian mathematical manuscript, there is a recipe for calculating the area of a circle by means of a procedure that amounts to using the

40 The recent proof of Fermat's last theorem by Andrew Wiles can be regarded as the latest and perhaps the most spectacular example of this fundamental idea. Fermat's last theorem is a purely arithmetic assertion, but its proof involves a tremendous amount of geometry, algebra and analysis.

fraction 256/81 as an approximation for π. In decimal form, one can check that 256/81 = 3.1605, which is a pretty decent approximation. (see Chace, 1929.) Nor was this approximation a purely empirical one. It was based on a piece of analysis which is illuminating. Accompanying the problem in which this approximation appears in the Rhind papyrus, there is a drawing that suggests that the area of the circle is being approximated by the area of an inscribed octagon. Thus, already one sees here the emergence of the idea that circular (or other) shapes could be approximated by polygonal shapes. In the Indian *Sulvasutras*, an approximation is derived by means of an elaborate geometric construction, which is tantamount to using the approximation $\pi = 3.088$.

Liu Hui (ca. 3rd century A.D.) was a Chinese mathematician who wrote a commentary on the *Nine Chapters.* In his commentary, Liu Hui gives two estimates for π, the first equal to 3.14 and the second equal to 3.1416; the first is correct to two decimal places, the second to four places. The method by which Liu Hui arrived at his estimate is very interesting. He observes that three times the diameter of a circle equals the perimeter of an inscribed regular hexagon, which is clearly less than the circumference of the circle. He therefore concludes that $\pi > 3$. Next he begins to approximate the area of a circle by the area of a regular polygon of $2n$ sides inscribed in the circle. He gives a heuristic argument as to why this approximation gets better as n becomes larger (thus showing that he had an intuitive grasp of the limiting process as it applied to this situation). Then, by calculating the areas of polygons of 6, 12, 24, 48 and 96 sides, he gets the estimate $\pi = 3.1416$, which is astonishingly accurate. The details of Liu Hui's procedure are given in van der Waerden (1983, p.197-199).[41] Another Chinese astronomer, Tsu Chhung-Chih (ca. 429-500 A.D.), obtained the estimate $\pi = 355/113$, which is even more accurate, by methods similar in spirit.[42] Both these show very clearly the clarity of thought and the imagination used by these early practitioners of the mathematical art. The estimate 3.1416 for π was also known to Apollonius of Greece and to the Indian mathematicians and astronomers Aryabhata and Bhaskara II.

41 The procedure that Liu Hui used was based on exactly the same idea as the one used by Archimedes about 500 years before. van der Waerden (1983) ascribes the method of Liu Hui to Apollonius and does not refer to Archimedes.

42 Professor Li Di informs me that the best of such estimates, made by Tsu Chung Chih and his son Tsu Geng, was: $3.1415926 < \pi < 3.1415927$. This estimate is accurate to six decimal places, and its accuracy was not surpassed for the next 1,000 years.

Indeed, van der Waerden (see van der Waerden, 1983, p. 212) believes that Apollonius was the source of all the other estimates. Whatever one's opinion regarding the issue of priority, one cannot fail to admire the imaginatively crafted solutions offered by these early mathematicians.

Along the same vein, one finds that the early mathematicians also devoted some thought to the calculation of areas of various plane shapes such as parallelograms, trapezoids, circles, etc., and volumes of various solids such as pyramids, spheres, cylinders, etc. Correct formulas for the areas of triangles, rectangles and trapezoids were known to essentially all early cultures.[43] I have discussed above the early attempts at determining the area of a circle in terms of its diameter or radius. Other plane figures such as ellipses do not seem to have attracted attention until a later phase.

On the other hand, turning to volumes, the ancient mathematicians in all cultures seem to have grappled with similar problems and came to very interesting conclusions. An early papyrus from Egypt, known as the Moscow papyrus (ca. 1850 B.C.), contains, as Problem 14, a correct formula for the volume of a truncated pyramid of a given height, and constructed on a square base. It says: "You are told: a truncated pyramid of 6 for the vertical height, by 4 on the base, by 2 on the top. You are to square the 4, result 16. You are to double 4 result 8. You are to square 2 result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right."[44] This is an instance of the following correct general formula

$$V = h(a^2 + ab + b^2)/3$$

for the volume of a truncated pyramid of height h, with the base being a square of side a, and the top a square of side b. It is not clear how this formula was derived, as no other Egyptian source refers to it in any way. The same problem was addressed in the

43 In the later Egyptian period, there occurs an *incorrect* formula for the area of an arbitrary quadrilateral, namely: it is asserted that if the quadrilateral has sides $a, b, c, d,$ then its area equals $(a + b)(c + d)/2$. This error was not rectified until the beginning of the Greek period in Europe. However, the Chinese and Indian mathematical traditions resolved the issue by dividing the quadrilateral into two triangles and calculating the areas of the two triangles separately. They realized that the formula is not so simple.

44 As quoted by Eves (1983, p. 41-42). See also Midonick (1965).

Nine Chapters in China, with the same (correct) solution. This formula is cited without proof in the original *Nine Chapters.* In his commentary on the *Nine Chapters,* Liu Hui provides a proof by using a dissection of the truncated pyramid into the following three types of solids: a rectangular block, a wedge with a rectangular base, and a pyramid with a square base, in which one of the edges is perpendicular to the base plane. The volumes of these are calculated separately, and the results added to give the correct formula

$$V = h(a^2 + ab + b^2)/3$$

for the volume of the truncated pyramid.[45]

The other two solids, namely the sphere and the cylinder, were considered by the Greek mathematician Archimedes, who was born in Syracuse in 287 B.C. It is likely that he spent some time at the famous University in Alexandria. Archimedes stands out as one of the giant minds of antiquity. His contributions both to physics and mathematics were manifold and uniformly magnificent.[46] In a work titled *Measurement of a Circle and Quadrature of a Parabola,* Archimedes gives a very accurate method of calculating π. He also investigated thoroughly the areas of parabolic segments. In doing these, his methods were astonishingly close to those of the Calculus. Namely, he dissected the region into approximating rectangles or triangles and took the sum of their areas as an approximation to the area sought. He then considered the limiting behavior of the approximating sums as the number of approximating rectangles grows without limit. Asserting correctly that the limiting quantity is the area of the circle, he finished that calculation. This is exactly the method followed by Newton 1,700 years later and by Riemann 200 years later still, the only difference being that these later mathematicians were able to add to the conclusion more rigorous proof resulting from developments in mathematics in the intervening centuries. In another work, entitled *On the Sphere and the Cylinder,* there appear theorems giving the area of the surface of a sphere and the volume of a sphere. In both cases, the area and volume are related to the surface area and the volume of the circumscribing cylinder. Archimedes showed, with a proof that

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45 See the very lucid article by D. B. Wagner (1979).

46 He contributed important results to geometry, number theory, mechanics and hydrostatics, astronomy, and geodesy. This we glean from the fragments of his writings that have survived. Many fragments have not.

would pass muster 2,000 years later, that the surface area of a sphere is equal to two-thirds of the surface area of a right cylinder that just circumscribes the sphere, and the volume of the sphere equals two-thirds of the volume of the same cylinder. The surface area and the volume of a cylinder are easy to compute, and this relation enables one to compute the surface area and the volume of a sphere quite easily.

In the *Nine Chapters,* there occurs a formula for the volume of a sphere of diameter *d*, which reads

$$S = (9/16)\, d^3$$

Of course, this is an approximation, and the value of π that is implicit in this approximation is $\pi = 3.375$. In his commentary on the *Nine Chapters,* Liu Hui not only explicates the method on which this approximation is based but goes much further. By using a principle known today as Cavalieri's principle (Cavalieri was a 17th century Italian mathematician)[47], he writes down a relation between the volume of the sphere and the volume of a solid that lies within the sphere. Liu Hui called this solid the "box-lid." The "box lid" is described as the solid that is common to two cylinders circumscribing the sphere, whose axes are perpendicular to one another. The relation between the two volumes was claimed to be

$$S = (\pi/4)\, B$$

where S is the volume of the sphere and B the volume of the box-lid. Liu Hui did not actually calculate the volume of the box-lid but left the matter in doubt, ending with a charmingly resigned poem about his inability to conclude the calculation and adding that "I dare to let the doubtful points stand, waiting for one who can expound them." The volume of the "box-lid" was calculated exactly by Tsu Keng-Chih some 250 years later. Again using Cavalieri's principle, completely correctly, Tsu Keng-Chih shows that

$$B = (2/3)\, d^3.$$

47 Cavalieri's principle states that two solids of equal height have equal volume, provided that their plane sections at each height have the same areas. Today we see that this principle follows from the integral calculus. As we mention in the text, Greek mathematicians showed a good grasp of it and used it with facility.

It is interesting to note that this relation, when combined with the relation between B and S given immediately before, yields

$$S = (\pi/6)\, d^{3} = (4\pi/3)\, r^{3}$$

which is the exact formula for the volume of a sphere.

Investigations of volumes of solids such as spheres do not seem to have been pursued in India with much success. The sources that have survived to our time either do not have any mention of such formulas (except for the case of a pyramid), or they use approximations that are much less exact than the ones that are found in Chinese or Greek sources. In a couple of cases, simply the wrong formula is recorded.

Interesting as these formulas for volumes of solids are, they do not represent the major achievement of Greek geometry. Indeed, the most significant achievement of Greek geometry was not any collection of formulas or facts, but a decisive change in point of view concerning the nature of geometry. This change in the point of view came from the way in which Euclid formulated Geometry. Euclid's formulation made it possible to view geometry as an axiomatic system, with a structure and an internal dynamic that was not driven by the exigencies of other walks of life. Previous to Euclid, Pythagoras had come close to this type of mathematics when his school studied the properties of various types of number, but even the Pythagoreans had a philosophical ax to grind when they studied numbers. Their study was often driven by their philosophical view that the world was organized by numerical laws, and the regularities and beauties of numbers provided a paradigm for the organization of the universe. With Euclid the situation changed in a basic way, developing in a conceptual direction stemming from Plato. Geometry became an abstract study to be pursued purely for the intellectual challenge and value it provided. Its applications, although important, were no longer necessary as the justification for the study of geometry.

Euclid began his formulation of Geometry with five *axioms* (statements, regarded as self-evident, which are to be accepted *a priori* without question). For instance, "The whole is greater than the part" is one of his axioms, as is the statement "If equals be added to equals, the wholes are equal" and so on. He had five such axioms followed by five postulates that asserted the basic properties of the ingredients of Euclid's geometry: points, lines, angles, circles, etc. Geometry now became the collection of correctly deduced assertions about these objects and their interrelations.[48]

Although Euclid lived and taught at the University of Alexandria, a center of learning in antiquity, the point of view that is contained implicitly in his approach did not become widely known or accepted (even moderately) until long after his time. Indeed, the Greek schools at Athens and Alexandria, at which followers of the early giants of Greek mathematics worked, had a troubled history more or less throughout their existence. The school at Alexandria suffered a disastrous setback when the library was burned in 48 B.C. by Julius Caesar's troops, and was eventually destroyed in 389 A.D. by an edict of the Emperor Theodosius, an edict inspired by the Christian Church's objections to the teachings of the school. The schools at both Athens and Alexandria struggled on until the middle of the 6th century, by which time they had become almost vestigial. The school at Athens was shut down in 529 A.D. under orders from the Emperor Justinian, again prompted by complaints from the Christian Church. The school at Alexandria remained open but relatively inactive until it was destroyed in the sack of Alexandria by the Muslims in 641 A.D. The next century would
see the emergence of West Asian scholars at the intellectual forefront of mathematics (and indeed in many other fields), a leadership position that they maintained for nearly six centuries.

Within a few years after the historic journey of Muhammad from Mecca to Medina in 622 A.D., Islam was a force to be reckoned with on the international scene. The Caliphs (as the rulers of the incipient empire were called) ruled from Medina and Kufa between 632 and 661 A.D. The period 661-750 A.D. saw the expansion of Arab rule under the Caliphs of the Umayyad dynasty. After 750 A.D., power passed to members of the ‘Abbasid family, who shifted the capital of the Caliphate to Baghdad. Within a century after that event, Arab hegemony had spread westward through North Africa into Spain, and the empire had a decidedly pan-Islamic (as distinct from an Arab identity). At first, Spain was ruled by the Caliph from the eastern capital, but after 755

48 One can say that Archimedes, who lived at around the same time as Euclid, represents a similar intellectual temperament. There is a lofty intellectual tone in his writings, and his insistence on clarity and rigor bespeaks a highly disciplined and rigorous view of mathematics. This point of view in writing about analysis was not to be appreciated among mathematicians in Europe for nearly 1,000 years. This approach to mathematics came back to Europe via West Asia.

A.D., there was a dual Caliphate, the original one in Baghdad (now called the eastern Caliphate) and the other one based in Cordoba in Spain, which now became known as the western Caliphate. They operated more or less independently of each other, with an unwritten acceptance of the superiority of the eastern Caliphate, and developed their own approaches to government and polity. The eastern Caliphate began to wane after the invasion of the eastern Caliphate by the Seljuk Turks in 1000 A.D., and thereafter continued to wane under the pressure of the crusades, which started about 1100 A.D. and lasted nearly 100 years. After the sack of Baghdad by the Mongols in 1258 A.D., the eastern Caliphate was on its last legs. The western Caliphate at Cordoba, on the other hand, prospered in the relative isolation of the Iberian peninsula and enjoyed its power till the end of the 15th century. Its hegemony ended when Cordoba fell to the Christians in 1492.

During the years between the 8th and 15th centuries, the court of the Caliphs became at once the patron, preserver, and propagator of the cultural heritage of Europe as well as India. In addition, the many scholars at the Caliph's court also contributed original ideas to the stream of the world's intellectual history. The former role was to prove crucial for the preservation and transmission of much that had been accomplished in Europe and India in the way of scientific knowledge. The Caliphs at Baghdad were great patrons of Science and the Arts. The court at Baghdad attracted many scholars, writers, etc., under whose guidance the Caliphs accomplished the translations of many Hindu, Persian and Greek works in astronomy, medicine and mathematics into Arabic. Indeed, in many cases, the Greek originals of works that were translated into Arabic later became lost to Europe until they were re-translated into Latin from their well-preserved Arabic translations. Through the agency of these scholars, Europe not only rediscovered its own culture after the dark ages, but also became aware of the extended boundaries of Asian thought and science.

Mathematical developments in the Umayyad and ‘Abbasid period (750 A.D. to 1250 A.D.) can be conveniently grouped in three categories:

(a) Extending the investigations of the past.
(b) Initiating new ideas, which would be later transmitted to mathematicians elsewhere.

(c) Preservation and transmission; in particular, translating and archiving important intellectual landmarks from Europe and Asia. I will briefly discuss each category.

(a) Extending the investigations of the past.

As mentioned above, the tradition of Euclid and Archimedes fell out of favor in Europe in the centuries immediately after their passing. Perhaps their ideas were just too advanced for their time. Euclid's deductive approach, which took a new view of geometry as an axiomatic system, found a fresh set of adherents among the scholars of the ‘Abbasid period. West Asian science always had a tradition in practical astronomy and navigation, and geometrical questions were perhaps natural in their context. Euclid's *Elements* were translated into Arabic. The first complete translation by Thabit ibn Qurra (ca. 826-901 A.D.), who also translated and commented on works by Apollonius, Archimedes and Ptolemy, led to an extensive study of the *Elements*. The new point of view that asserted the possibility of geometry being an axiomatic system seems to have excited the Arab imagination mightily. The *Elements* were studied assiduously, and there followed an intensive study of the axiomatic structure of the Euclidean system, a study to which there are significant documented contributions by at least three scholars: the Persian mathematicians Abu al-Wafa’ (ca. 940-998 A.D.), ‘Umar al-Khayyam (Omar Khayyam ca. 1048 - 1131 A.D.) and Nasir al-Din al-Tusi (ca. 1201-1274 A.D.).

Their work on Euclid's Parallel Postulate is a landmark in the history of mathematics. The Parallel Postulate is the last postulate in Euclid's *Elements* and can be stated as follows: given a line *l* and a point P not on *l,* there is one and only one line through P which is parallel to *l.* Ever since the formulation of this postulate, there had been debate on the issue of its status as a postulate. The question was whether the postulate was redundant; i.e., whether it was a consequence of the other postulates in Euclid. This debate occupied mathematicians for several centuries. It was only in the 19th century that the debate was settled, with the discovery of non-Euclidean geometries by Lobachevski, Gauss and Bolyai. The parallel postulate was shown to be independent of the other postulates in Euclid.

Abu al-Wafa', Omar Khayyam, and Nasir al-Din al-Tusi all wrote extensively on the problem of the independence of Euclid's parallel postulate. In doing so, they explained in more detail many aspects that were left implicit in the *Elements* and clarified many misconceptions about the structure of axiomatic systems. Indeed, it seems clear that their writings were essentially the first ones that show a critical understanding of the nature of such systems, and especially of the notion of independence in such systems. Their work, eventually translated and taken back to Europe, was quite influential in molding the modes of thought of European scholars. For instance, the discussions of these scholars concerning the parallel postulate directly influenced Saccheri (1667-1733), who was familiar with Nasir al-Din al-Tusi's work. Saccheri's later work on non-Euclidean geometry was inspired directly by the questions that Nasir al-Din al-Tusi had commented upon. John Wallis, the 17th century British mathematician, translated Nasir al-Din al-Tusi's writings into Latin (cf. Eves, 1983, p. 172), and lectured on them at Oxford.

The issue of the independence of the Parallel Postulate continued to engage some of the best mathematicians of Europe for several hundred years. Many incorrect proofs were proposed purporting to show that the Parallel Postulate was in fact a consequence of the other postulates. In the 19th century, Bolyai, Gauss and Lobachevsky independently arrived at a proof of the independence of the Parallel Postulate by constructing a geometric model in which the other postulates were verifiable to be true, while the Parallel Postulate was verifiably false. In turn, their work led to further work by others — notably Beltrami, Klein, Poincare and Hilbert — concerning the nature of geometrical systems, leading to the formulation of fundamental philosophical questions such as completeness and decidability of axiomatic systems. The epochal work of Goedel, in the middle of the present century, concerning the incompleteness of *any* axiomatic system with a finite set of primitive notions and axioms, is in the same lineage of intellectual issues. Thus, one sees a clear thread running from issues addressed by the scholars of the 'Abbasid period to some of the most profound epistemological issues of our day.[49]

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49 See Rosenfeld (1988) for more information concerning these issues.

There are other instances, of varying degrees of significance, of mathematical investigations by the scholars of the ‘Abbasid period which extended and refined earlier investigations, especially by Greek and Indian mathematicians. Such refinements had profound influences on the work of later European scholars and left an indelible imprint on the world's intellectual history.

(b) Initiating new ideas.

One of most interesting mathematical contributions of this period was the use of geometric methods to solve an algebraic problem. The essence of this method is that an algebraic problem is first translated as a geometric problem involving one or more curves, and it is shown that determining certain geometric information about them (e.g., their points of intersection or the location of certain tangents) leads to the solution of the original algebraic problem.

The mathematician Thabit ibn Qurra (a Syrian Semite who hailed from Harran in present day Turkey) authored a remarkable treatise called *On the Verification of the Problems of Algebra by Geometrical Proofs* in which he showed how one may solve a wide variety of quadratic equations by geometrical methods. Thabit ibn Qurra's influence remained an inspiration to those that followed him and culminated in further advances of the same kind. One of the best known such results is due to the Persian mathematician, ‘Umar al-Khayyam (Omar Khayyam). He was able to show by a purely geometric construction how one can find the roots of a *cubic* equation of the form

$$x^3 + b^2x + a^3 = cx^2$$

where a, b, c are positive numbers.[50] The construction is sophisticated, requires a significant preparatory procedure, and shows a firm command of the techniques of what began to be called "Geometric Algebra" many years later. The notion that one could use geometric constructions for certain types of algebraic problems was certainly recognized by Euclid and Archimedes, but before Omar Khayyam's construction, only simple types of equations — namely, linear equations or some quadratic equations —

50 Such an equation always has at least one positive root. This assumption is necessary for Khayyam's geometric construction to yield a meaningful result.

were thought to be amenable to the geometric method. By giving an example of a third degree equation that could be solved in this way, Omar Khayyam opened the door to the study of the more general question: What kind of algebraic problems can be represented and solved successfully in this manner? Omar Khayyam's inspiration probably came from the work of another Persian mathematician, Abu al Wafa', who had earlier used geometric methods to find solutions of certain special quartic equations (equations of the fourth degree). Khayyam's realization that such methods are powerful enough to apply to a whole class of cases systematically is an important idea, which gave rise to many a later development.

(c) Preservation and transmission.

After the establishment of the Caliphate, there was a period of expansion and consolidation of the empire that came to a close by the end of the 8th century. In the years immediately following, there was a spurt of activity by diverse scholars from Persia, Iraq, Turkey and other parts of the empire, who avidly collected important scientific works of various kinds from different parts of the world (with whom the conquests of the preceding century had familiarized them) and devoted a great deal of attention to translating and assimilating their contents. Notable among mathematical works are translations of the work of Aryabhata and Brahmagupta from India (ca. 766 A.D.)[51] and of Euclid, Archimedes, Apollonius, Pappus, Ptolemy, Aristotle from Greece. Also notable is the careful archiving of several tables, such as tables of multiplication, of square roots and cube roots, and (later) of trigonometric functions. This list is but a small fraction of the total activity that occupied the scholars at court.[52] Three successive Caliphs (al-Mansur, Harun al-Rashid, and al-Ma'mun), their reigns spanning the period 755-833 A.D., actively encouraged the importation of manuscripts and their study and provided patronage at court for a large number of scholars who were engaged in their translation and analysis. Moreover, by the end of this period,

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51 Many scholars believe that the Hindu-Arabic numerals came to Western Asia at this time and were later propagated to Europe through Islamic mathematicians. I have referred above to al-Kwarizmi's treatise about their nature and use.

52 I mention here only the Eastern Caliphate. One must note that there was also an appreciable amount of scholarly activity in Cordoba, Seville and Toledo, inspired by the support accorded to scholarship by the Western Caliphate. Scholars there came from many parts of Europe and North Africa, as well as from the eastern reaches of the Arab Empire.

although succeeding Caliphs were not as supportive of scientific activity as these three Caliphs, a tradition of patronizing scientists at court had been firmly established and was to continue well into the later years of the Caliphate. Later scholars such as al-Biruni (second half of the 10th century A.D.), Abu Kamil (ca. 900 A.D.), and al-Karkhi (late 10th century A.D.), also played their part in this process. Many works by Greek mathematicians (e.g., Diophantus, Theodosius, Heron) were translated in this epoch. In many cases, the translations proved to be crucial for the later development of European science. Indeed, the original versions of many Greek works were swept into oblivion by the tide of change that drove European culture in the dark ages, and European scholars came to know of the existence of the originals only when they later became aware of their Arabic translations. These Arabic translations were then re-translated into Latin. Several Greek works became known to European scholars only through this process, which was facilitated by increased direct contact between Europe and the eastern Caliphate and its successors, or via Spain, which became an influential point of contact between West Asian and European scholars.

In assessing the mathematical contribution of the scholars of the Umayyad and ‘Abbasid period, the discussion by European scholars often tends to be concentrated on the role played by them in preserving and transmitting knowledge from the Greek and Indian traditions. Such assessments often relegate those scholars to the status of chroniclers, and the nature of their effort to routine record-keeping. In my opinion, this view shows a non-understanding of the processes that keep alive a tradition that survives a long time. It seems to me unlikely that the interest of a large community of scholars could be sustained over centuries if they were acting merely as scribes. Indeed, they would necessarily have to be engaged actively in the issues arising from the manuscripts they were trying to understand and preserve. There is plenty of evidence that the mathematicians of this period were in fact actively engaged in mathematical issues and made a number of original contributions. However, as political and scientific hegemony passed into Europe, the significance of their contributions perhaps receded from the consciousness of European historians of science to greater degree than it need have.

4.4 Surveying, astronomy, cartography.

Much of the mathematical activity in antiquity was centered on mensuration, geodesy, astronomy, and navigation. The mathematics that these subjects naturally require developed from the applications themselves. Trigonometry and spherical geometry were two branches of mathematics that benefited from this circumstance.

Almost all ancient cultures were concerned with elementary problems of mensuration (measurement of different shapes), especially of land. Formulas and procedures for the calculation of simple areas probably grew out of this concern. I have discussed many of those problems above. A different set of problems arise from the need to demarcate or measure physical features in the process of construction. One sees all over the ancient world many feats of large scale construction — aqueducts, roads, tunnels, temples, etc. — that would have been impossible without accurate methods of surveying. Often, one cannot tell accurately what mathematical procedures were used in antiquity, due to the fact that the historical record may be too scanty. Here as in other areas one has to piece together the picture from such record as survives to the present time.

One of the earliest records of a systematic method of treating the sort of problems that come up in surveying is the 3rd century work of Liu Hui entitled *Hai-tao suan-ching (The Sea Island Mathematics Manual).* The following problem is posed:

> Someone wants to measure an island. Two poles are erected, both 30 feet high, one nearer to the island, the other farther away, in a distance of 1000 paces (1 pace = 6 feet). The pole that is further away is exactly in one straight line with the first one and the island. If the eye looks towards the top of first pole from the earth at a distance of 123 paces, it just sees the highest point of the island. If one places oneself in the same way 127 paces behind the other pole, one sees the peak of the island on the visual ray passing from the earth to the top of the second pole. To find the height of the island and its distance. [53]

53 See van der Waerden (1983, p.193); for further information on the *Sea Island Manual*, see Swetz (1992).

Figures 6a and 6b illustrate the geometry of the problem posed.

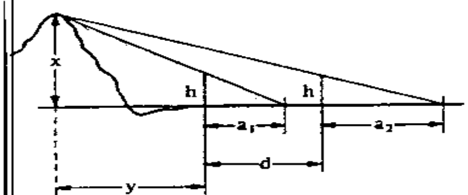
Liu Hui proceeds to give a solution to the problem by describing step-by-step the operations that would be needed to compute the solution. In the figure below, the height of the island peak is labeled as x, and the height of the poles h. The other quantities are labeled as shown, by the letters y, d, a_1, a_2, *etc.*

The solution is

$$x = hd / (a_2 - a_1) + h$$

$$y = a_1 d / (a_2 - a_1).$$

It is clear that Liu Hui was familiar with the properties of similar triangles, especially the fact that in two similar triangles, the ratios of corresponding sides are always equal. Except for the fact that trigonometric functions are not used by name in this problem, the approach (via similar triangles) is the same as the one that underlies surveying even in our day. Here then is a technique that has not been essentially modified ever since it was discovered. Liu Hui continues with a number of problems in the *Sea Island Mathematical Manual*, all based on the same mathematical principle, which has been called “The Method of Double Differences.” The rigorous manner in which the technique is expounded leaves no doubt that Liu Hui was on the edge of formulating trigonometric functions, and would probably have formulated them more precisely if his researches had led him in directions that needed the formulation. While there is no record of similar procedures being used by other cultures for the same kind of problem, the elementary nature of the method, as well as the practical need to have such a tool available in the construction of large structures, both argue that some form of procedure resembling this was probably well-known in many cultures of the ancient world. As van der Waerden remarks (1983, p. 192), where he discusses Liu Hui and Aryabhata, "with Liu Hui we leave the domain of popular mathematics. His geometry is on the same level as the geometry of Euclid; he presents proofs of really difficult theorems. Aryabhata gives no proofs, but his trigonometry and mathematical astronomy are by no means elementary." In India, astronomy had been highly cultivated both for scientific and ritual or astrological motives. By the end of the 6th



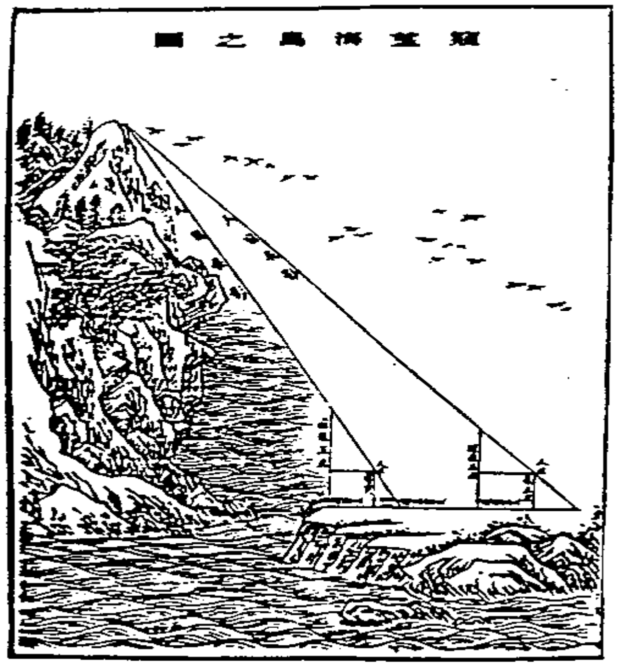


Figure 6a
Schematic from van der Waerden (1983, p.193)

Figure 6b
Picture from *Thu Shu Chi Chheng* (Imperial Encyclopedia, 1726 A.D.) Attributed there to Liu Hui's *Hai Tao Suan Ching* (263 A.D.) As shown in Needham (1959, p. 572)

Figure 6
The Method of Double Differences

century, the Indian astronomer Aryabhata had introduced the trigonometric function that is today called the sine function, and had compiled a table of sines, wherein a quadrant of the circle is divided into 24 equal parts. The table of sines is reasonably accurate.[54] In van der Waerden (1983, pp. 208-213), there is an extensive discussion of Aryabhata's method. It is too technical for the purpose of this essay. The conclusion drawn by van der Waerden is that Aryabhata probably got his inspiration from the Greek geometer Apollonius and must have had access to the latter's work.

In West Asia during the period of the Caliphs, similar trigonometric studies were undertaken. The Persian mathematician Abu al-Wafa', whom I have mentioned previously, was fully conversant with the trigonometric functions known today as the sine and cosine. Moreover, he seems to have been the first to define the function which is called the tangent. Not only did he define this function, he also compiled a table of sines and tangents, by 15' intervals, which was accurate to about 1%. Later, in the 15th century, Ulugh Beg, a Mongol who was at that time the ruler of a Persian province, and who was also an astronomer of considerable ability, compiled a table of sines and tangents for angles, progressing by 1' of arc. These were remarkably accurate, namely to eight decimal places. This would be a computational feat in any age. Given the limitations under which Ulugh Beg must have worked, it is an astonishing feat of computation. No doubt the compiler must have known quite a lot about the properties of the trigonometric functions, for, without a thorough understanding of their properties, it would be impossible to compile such an extensive and accurate table.[55]

In the realm of astronomy, the contributions of the ancient astronomers have been extensively studied, and thanks to the monumental works of Otto Neugebauer — of which Neugebauer (1975) is just a small sample — we have a very good idea of the sophistication of which ancient astronomers were capable.[56] They were able to predict the onset and duration of eclipses, had an excellent map of the night sky in the northern

54 The Indian table of sines was known in China by the 8th century A.D. Moreover, Zhang Sui, an 8th century Chinese astronomer, also compiled a table of tangents. I owe this information to Professor Li Di. See also the article by C. Collen in Sivin (1970- , v.5, pp. 1-34).

55 For more information on the contributions of the scholars of the 'Abbasid period to trigonometry, see Kennedy (1969) and Kennedy (1983).

56 Astronomy was also highly developed in the ancient cultures of South America. The interested reader is referred to the works of Aveni (1975, 1980, 1989, 1992).

hemisphere, and had a very accurate understanding of the positions of the planets in the night sky at various times of the year. The level of detail is quite staggering and persuades us that the ancients were not only patient and accurate observers, but also must have had adequate theoretical (perhaps partly heuristic) tools with which to organize and use such detailed data for astronomical prediction. Exactly what level of theoretical understanding they reached is not known to us, due to the lack of surviving records, but reasonable inferences can be drawn from the results they obtained.

In the works of Aryabhata, there is a mixture of trigonometric techniques with certain techniques from the realm of number theory. Specifically, during the course of his investigations, Aryabhata comes upon a situation in which one needs to solve a linear equation of the form

$$mx + ny = l$$

where m, n, and l are integers. The problem is whether one can find solutions x and y such that these are *integers*. Such problems are a part of the study of the theory of numbers and are called Diophantine problems, after the Greek mathematician Diophantus who studied a wide class of such problems. The method devised by Aryabhata is a precursor of the methods employed by number theorists 1,200 years later. Aryabhata calls his method the "pulverizer" because it "decimates" the problem by reducing the coefficients to ever smaller numbers, thus "pulverizing" the difficulty of the problem. The method can be described *roughly* as follows: starting with an equation such as $mx + ny = l$, where $l, m, n,$ are given integers, and values for x and y must be found, it is shown how the solution can be affected by solving another similar equation of the form $ax + by = c$, where, however, the numbers $a, b, c,$ are *smaller* than $l, m, n.$ By repeatedly applying this method, the problem is reduced to a more manageable one, in which the coefficients are smaller.

The idea used by Aryabhata is very similar in spirit to the idea of "descent" used by Fermat and Lagrange nearly a thousand years later. It is interesting to take note of the power of the methods used by Aryabhata by considering the size of calculations that he was able to accomplish by using these methods. In the ancient Indian system, all heavenly bodies were assumed to have been in alignment at the beginning of an astronomical age, called a *kalpa.* After a certain period, called a *mahayuga,* they would return to the same positions. One thousand such *mahayugas* were assumed to constitute the natural life of the universe. Using a period of 4,320,000 sidereal years as the fundamental return period of all planets (a period he called the *mahayuga,* the great era), he calculates the number of revolutions made during this period by each heavenly body, including the moon and 5 planets. Also the apogee of each body in its orbit and some other data are computed. The calculations, which have been thoroughly researched by modern scholars, are mathematically quite sophisticated (see, for example van der Waerden, 1983, pp.122-131), and the periods of the planets calculated therein are in excellent agreement with present day calculations, with an error of less than 1%.[57]

The methods of Aryabhata and Brahmagupta are theoretically significant. In an exhaustive discussion, van der Waerden (1983, p.122 ff.) very clearly elaborates the exact nature of Aryabhata's contribution. His account makes clear that already by the 6th century, there had been great progress made by Indian astronomers in the computation of astronomical orbits. According to van der Waerden, the influence of Hellenistic ideas is evident. Whatever one's opinion of the extent of this influence (it is certain that there was some), the main feature of the Indian calculations is the development of new theoretical tools (the "pulverizer"), the further development of old tools conceived by Diophantus, and the much greater accuracy to which the calculations were carried. I shall not go into greater detail on so technical a subject.

One final topic that is worth mentioning in this context is the impetus provided by astronomical cartography to the study of spherical trigonometry. Mapping of the night sky accurately requires some study of the latter subject. Very little in the way of a record has survived to the present, but one can gauge the sophistication attained by the astronomers of the period of the Caliphate by a couple of fragments that are available. The law of cosines for a spherical triangle

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

was given by al-Battani (ca. 880 A.D.), who also gave a related formula

$$\cos B = \cos b \sin A$$

57 If we recall the fact that a *kalpa* was 4,320,000,000 years long, we can appreciate the size of the numbers that the Hindu numeral system enabled these astronomers to handle with accuracy and ease.

See Figure 7 for an identification of the various symbols that occur in these equations.

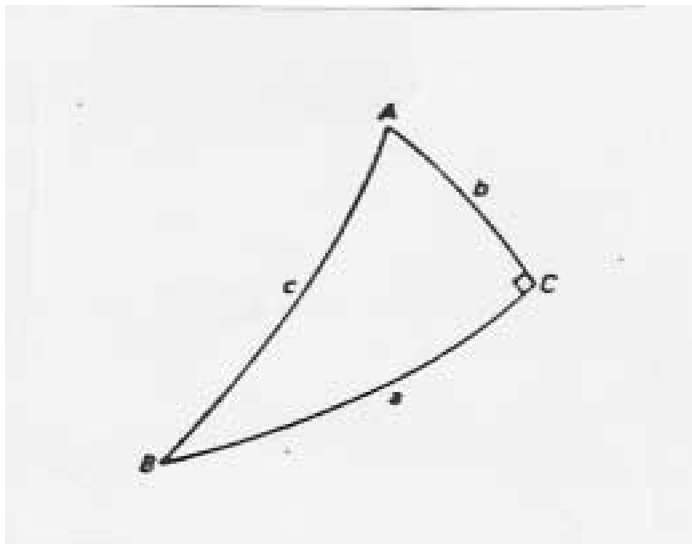


Figure 7
Spherical triangles

A peripheral index of the impact of the ideas of Indian and West Asian scholars on astronomy and trigonometry is the etymology of a number of technical terms in these two fields that are in common use today. A large number of names of stars — e.g., Aldebaran, Vega, Rigel — can be traced back to Arabic words which were used as translations of phrases in Ptolemy's Almagest, the single most influential work of ancient astronomy. In this connection, Eves (1983) gives the following examples:

> Thus we have Betelgeuse (Armpit of the Central One), Fomalhaut (Mouth of the Fish), Deneb (Tail of the Bird), Rigel (Leg of the Giant), and so forth.

The origin of the word sine is curious. Aryabhata called it *Ardha-jya* (half-chord) and also *jya-ardha* (chord-half), and then abbreviated the term by simply using *jya* (chord). From *jya* the Arabs phonetically derived *jiba*, which, following Arabian practice of omitting vowels, was written as *jb*. Now *jiba*, aside from its technical significance, is a meaningless word in Arabic. Later writers, coming across *jb* as an abbreviation for the meaningless word *jiba*, substituted *jaib* instead, which contains the same letters, and is a good Arabic word meaning "cove" or "bay". Still later, Gherardo of Cremona (ca. 1150 A.D.), when he made his translations from the Arabic, replaced the Arabian *jaib* by its Latin equivalent *sinus*, whence came our present day word *sine*.[58]

4.5 Art and architecture.

Art has an appeal that often transcends cultural boundaries. The plastic arts, such as sculpture and architecture, and the performing arts, such as music, speak to us in mysterious and universal modes. There are visible interactions between mathematics on the one hand and the fine arts on the other. In this section, I briefly describe the role that geometry played in the art and architecture in certain Asian cultures in the past. Reflecting on these connections can provide us with a perspective on the nature and the direction of the interaction between mathematics and the arts.

Geometry and art are both concerned with the understanding and organization of space. Therefore, it is not surprising that natural connections should exist between the two disciplines. Two aspects of art and architecture have had particularly significant interactions with geometrical ideas. The first is the design of two-dimensional repeating patterns, which one finds in many buildings in various parts of the world. The second is the treatment of perspective in paintings.

The treatment of perspective is an achievement of the Renaissance artists in Italy and was refined by other traditions following them, chiefly in Europe. Cultures in other parts of the world do not seem to have been concerned with the notion of perspective in early times. Although there are a couple of isolated examples of the use of perspective, such as in the work of Ying Zao Fa Shi (11th century A.D.) in China, it seems clear that the treatment of perspective was not a primary concern. Perhaps because Symbolism rather than Realism provided the primary impetus to art in these

58 Professor S. Nomanul Haq has pointed out to me that the explanation of the Arabic word *jaib* in this quotation is not clear. He suggests an alternative etymological sketch, which I prefer: Sanskrit *Ardha-jya* (half-chord) → *jya* (chord) → Arabic *jyb* (pocket) → Latin *sinus* → English *sine.*

cultures, the issue of perspective was not explored very far by the artists in these cultures. On the other hand, Realism was clearly a major artistic goal of the Italian artists of the Renaissance (and especially of Leonardo da Vinci), who explored and effectively mastered the techniques of perspective. These techniques then diffused from the Italian school to other traditions, in Europe or elsewhere.[59]

In the context of architectural design, however, one finds many examples of mathematically interesting patterns being used, mainly in the architecture of the West Asia, but occasionally elsewhere as well. In Figures 8a and 8b I have reproduced a few interesting patterns from the books by Bourgoin (1971) and Dye (1981). In examining these, it can be noticed that (a) the patterns are repetitive (more precisely, they consist of congruent replications of a shape); and (b) if one imagines them repeated endlessly in all directions on the plane of the paper, they fill up all the space available, without gaps or overlaps. Patterns in a plane which have these two properties are called regular tilings (or tessellations) of the plane. The geometric principles behind generating such patterns are well understood today and form the basis of a number of algorithms that can be used for the computer generation of such patterns. In the historical instances in which these patterns occur, these principles were not, of course, formalized in a mathematically cogent way, but the tremendous variety of tiling patterns that are found in China and in many parts of Central and West Asia suggests that those who designed them had an excellent practical understanding of the manner in which such tilings could be generated. From that understanding they were able to extract a number of unusual patterns using purely experimental methods. From the perspective of present day mathematics, it is known that the theories that fully explain the basic structure of such tilings depend on what is today called group theory, a part of abstract algebra, the fundamentals of which were formulated only at the end of the 19th century.

59 Perspective is of course a very good example of Geometry applied to Art. The development of techniques of treating it realistically was followed in European Art by reactive styles in which the perspective was either treated in a way that borders on realism but contains carefully designed but subtle accents that are not in strict accord with reality (as we see in the work of Edward Hopper, for example), or was sometimes abandoned altogether (as for example in the work of Braque or Picasso).

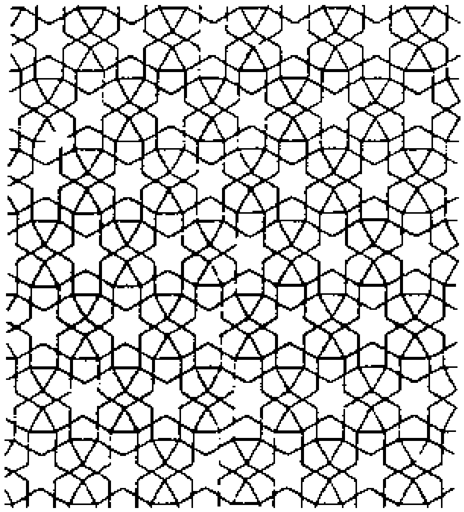


Figure 8a
An Arabic pattern
From Bourgoin (1973, p.4)

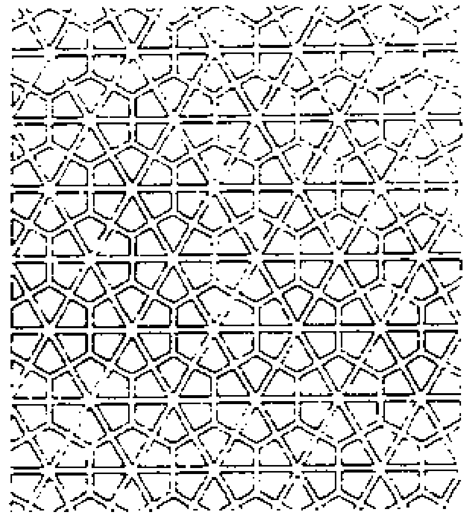


Figure 8b
A Chinese pattern
From Dye (1981, p.16)

In order to get a better feel for the type of devices employed by these designers, let me examine a couple of specific examples of the types of tilings illustrated in Figures 8a and 8b in a little more detail.[60]

60 For a very readable, more detailed, account, see Williams (1993), which treats these topics.

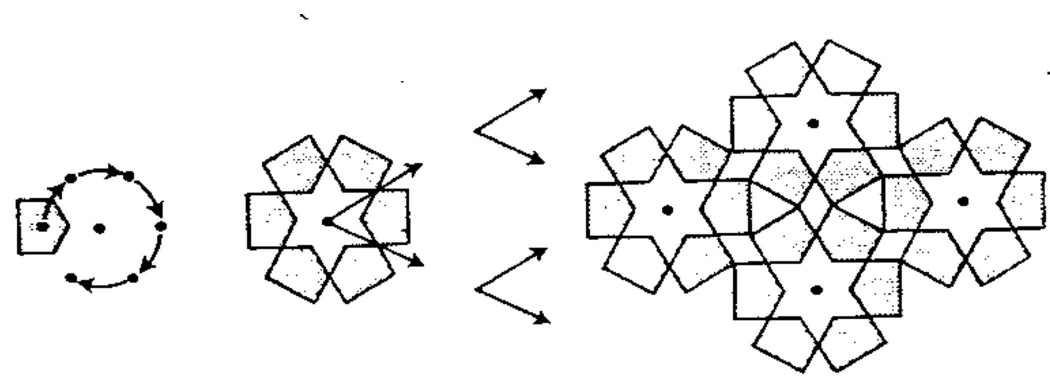
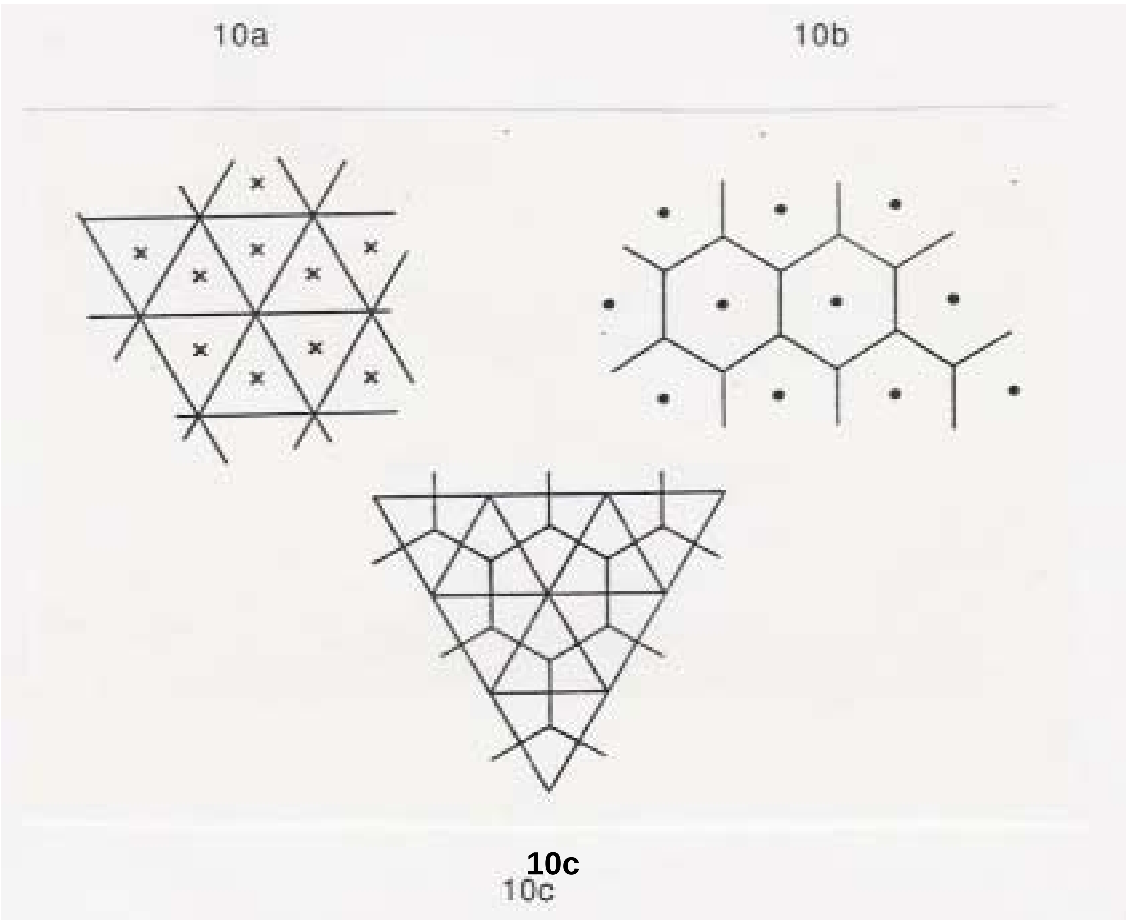


Figure 9
Pattern generation by rotation and replication
From Williams (1993, p. 157)

First consider the pattern in Figure 8a. Start with the basic wedge-like shape on the extreme left in Figure 9. One notes that by rotating that shape successively around the center of the circle shown, one gets the middle configuration of Figure 9.

Repeating this configuration by duplication, one gets the configuration on the extreme right in Figure 9, which produces the pattern of Figure 8a. Notice that the basic shape in the middle in Figure 9 does not by itself fill all the space available on the paper. Rather, there appear triangles and rhombuses (shaded in Figure 9), more or less automatically. The resulting pattern fills up all of the plane. The procedure given here is in two parts. First, a basic building block is generated, as in Figure 9, by using the implicit notion of rotational symmetry. More precisely, that building block has rotational symmetry of a sort, in the sense that if one rotates that figure around its center by a sixth of a circle, the figure suffers no visible change. One may roughly call this six-fold rotational symmetry. Next, replication is used, in the sense that a copy of the same basic figure is simply transported to another spot in the plane, in such a way that the copy or copies of the original figure fit nicely together. Doing this a number of times, it is clear that the whole plane can be filled with this pattern.

The Chinese pattern of Figure 8b can be understood better by considering Figure 10.



10c

Figure 10
Generation of a pattern by superposition
From Williams (1993, p. 149)

One can look at the pattern of Figure 8b in two illuminating ways: First, notice that if one looks just at the centers of the hexagons that appear in the pattern, those centers form the vertices of equilateral triangles that fill out the whole plane. On the other hand, the hexagons themselves also form a pattern that fills out the whole plane. Thus, one can regard the pattern in Figure 8b as a pattern obtained by combining and superposing these two plane-filling patterns. Now consider these observations in the context of Figure 10. At the left in the top set of figures is the triangular grid that can fill the plane (Figure 10a). On the right, also in the top row, is the hexagonal pattern (Figure 10b). By superimposing the two in such way that the corners of the triangles in the former pattern fall exactly at the centers of the hexagons in the latter pattern, one gets the grid that is the pattern of Figure 10c, which is the same as the grid in Figure 8b.

Here is an alternative construction: Notice that the basic hexagons in Figure 8 appear to be divided into six kite-like figures (see Figure 11).

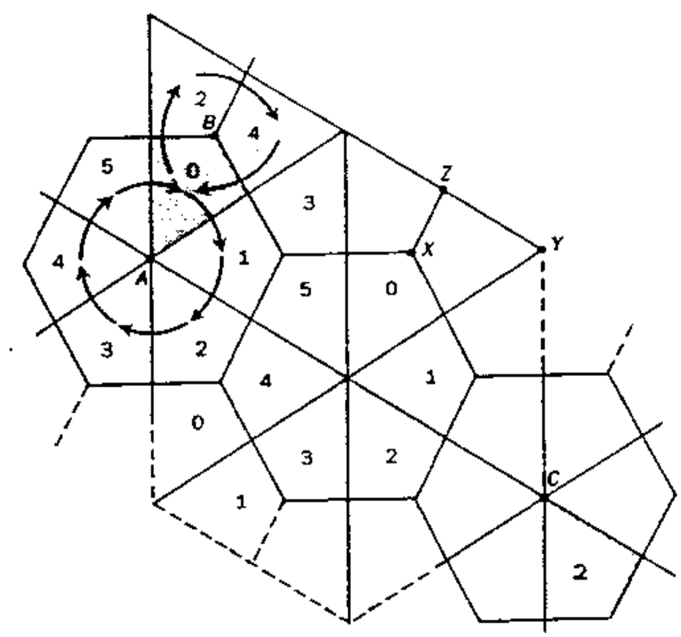


Figure 11
Generation of a pattern by rotation around different pivots
From Williams (1993, p.153)

One can therefore *start* with this kite-like shape, marked as the shaded piece with the numeral 0 in Figure 11, and rotate it around the vertex A, successively to the five new positions marked as 1, 2, 3, 4 and 5, generating the hexagon. On the other hand, the same kite-like figure can also be rotated around the vertex B (instead of the vertex A), yielding two new positions around B, marked by the numerals 2 and 4 in Figure 11. One can repeat this procedure, as follows: Each kite-like figure has a "narrow" end, like A, and a "broad" end, like B.[61] By rotating each kite-like figure around the narrow end, one generates a hexagonal configuration of kites, and by rotating it around the broad end, one obtains a triangular configuration. Thus, the entire pattern can be thought of as having been generated by these successive positions of the basic kite from which one started, and all of its offspring (of *all* succeeding generations) obtained by the process of successive rotations as described above.

One can see easily from these two examples that something very interesting is going on here. The general question that one might pose is: Can one describe a systematic way by which *all* such tilings (with straight sides, and having a repetitive structure) could be obtained? If so, how? The study of these questions forms a fascinating part of modern mathematics, in which the notions of group theory come into play. While the architectural patterns that one sees in Islamic and Chinese architecture did not base themselves on such elaborate theories, their intuitive grasp of the symmetries of the pattern is surely interesting.

As a coda to this section, I mention two items which might be of interest to the reader. The first is that although all the tilings considered above are tilings in which the basic shapes are regular polygons bounded by straight lines, it is possible to construct tilings where the basic shapes are not subject to this restriction. Such tilings are illustrated in Figures 12a, b, and c. Notice that the boundaries may be curved, as in Figure 12a, or the shape may be irregular, as in Figure 12b and 12c.

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61 We can be more precise here. The "narrow" end in the kite-like figure has an angle of 60 degrees; the "broad" end has an angle of 120 degrees. The other two angles are right angles, of 90 degrees each.

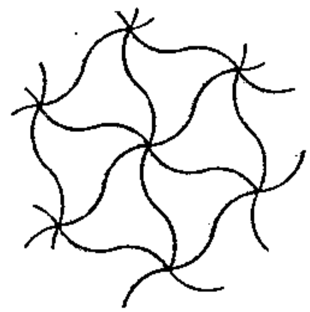

Figure 12a
Tilings with polygons with curved boundaries
From Williams (1993, pp. 162-63)

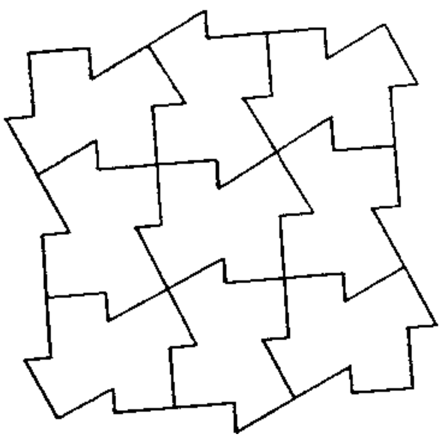

Figure 12b
From Williams (1993, pp. 162-63)

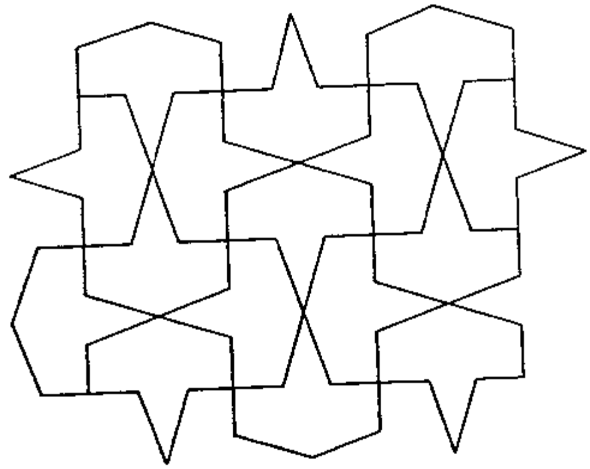

Figure 12c
A design on a tomb tower
Kharraqan, Iran (ca. 1459 A.D.)
From El-Said & Parman (1976, p.19)

Figure 12
Tilings with irregular polygons

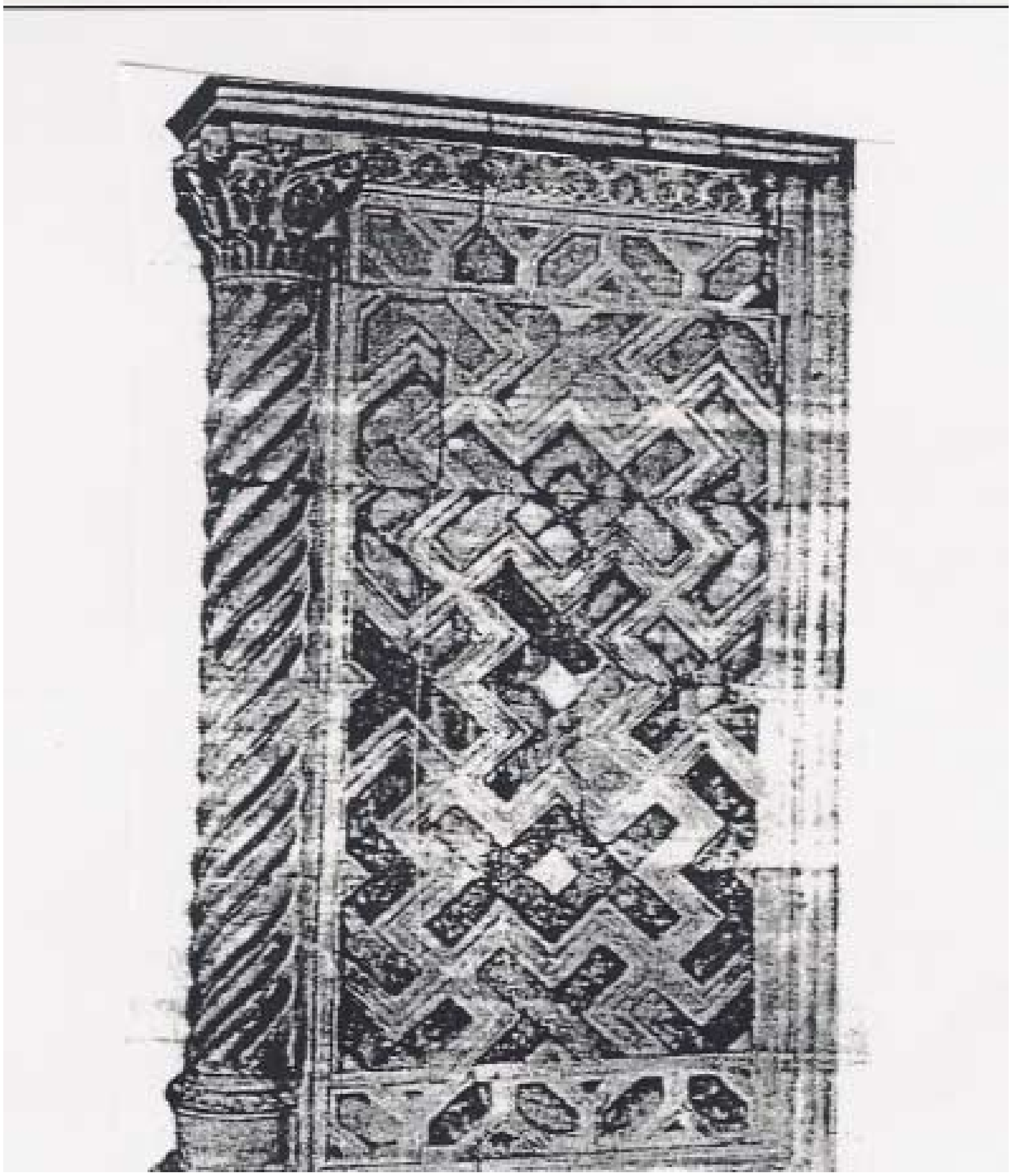

Figure 13
A thirteenth century portal from Konya, Turkey
As shown in El-Said & Parman (1976, p. 16)

An actual architectural use of patterns which use L-shaped motifs is shown in Figure 13.

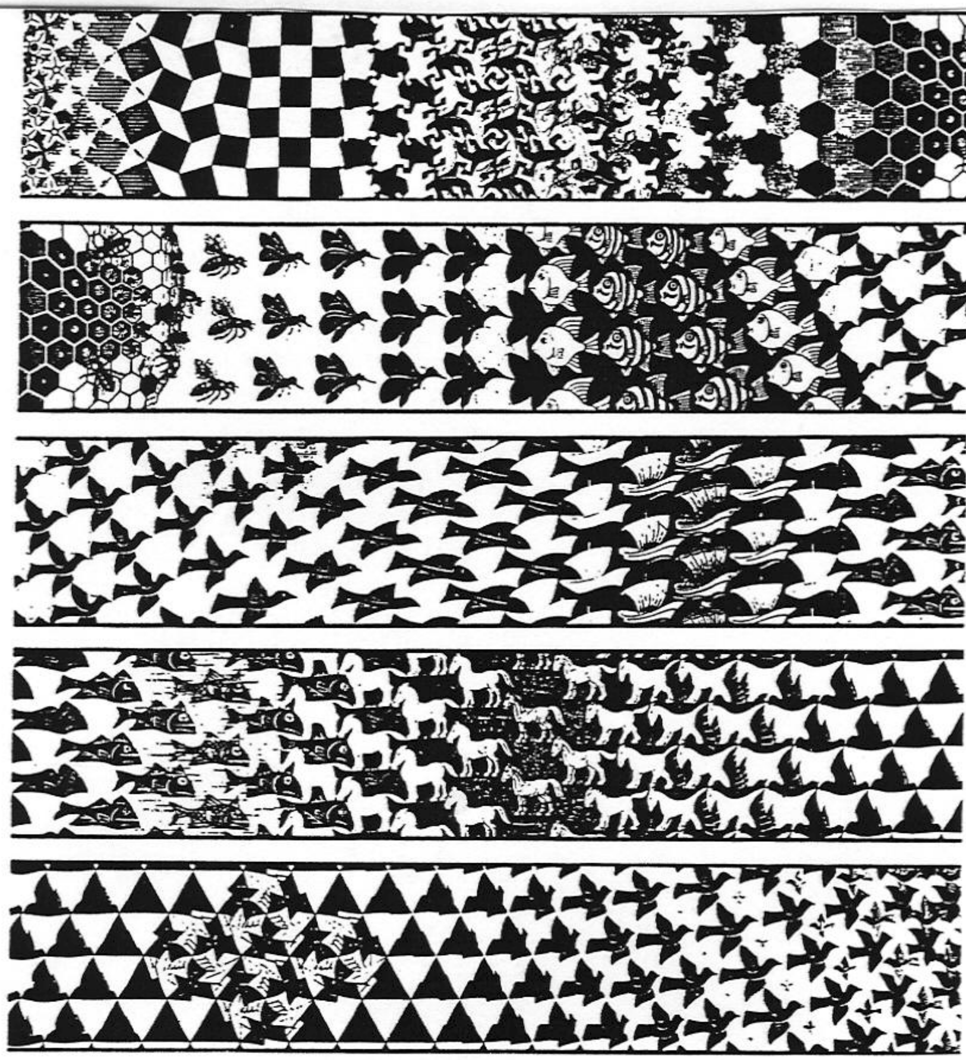

Figure 14
Some tilings used by M. Escher
From Escher (1981, p. 32)

Finally, for a beautiful set of examples of such repetitive tilings, which nevertheless are not built up of simple shapes with straight boundaries, one can look at the art of M. C. Escher, the celebrated Dutch artist. The collection Escher (1981) has some splendid examples of the ingenious tilings, utilizing a combination of geometric shapes with shapes of birds and animals. See Figure 14 for a sample.

The second point is that recently, the famous British mathematician and physicist Sir Roger Penrose has discovered several tilings of the plane (called Penrose tilings) which are *not repetitive* in the sense that they cannot be conceived of as being obtained by replicating a piece into congruent offspring over and over again. These tilings *appear* as if they have some repetitive structure (periodicity), but they have subtle departures from periodicity which make them what mathematicians have called *aperiodic* tilings (meaning that they do not periodically repeat themselves). A specimen is shown in Figure 15. Grunbaum & Shepherd (1986) give a detailed mathematical treatment of such tilings. Surprisingly, some of these aperiodic patterns occur in nature because they govern the formation of what are called quasi-crystals by chemists. These materials have a structure that is *almost,* but not quite, crystalline, and sometimes have exotic properties as materials. I cannot go into more detail here, but Stewart & Golubitsky (1992) offer a non-technical explanation that might interest the reader.

I close this section with the following thought: Is it not a beautiful illustration of the continuity and power of mathematical ideas that such a connection can be made between ancient architectural patterns on the one hand and quasi-crystalline materials that have exotic physical properties on the other hand?

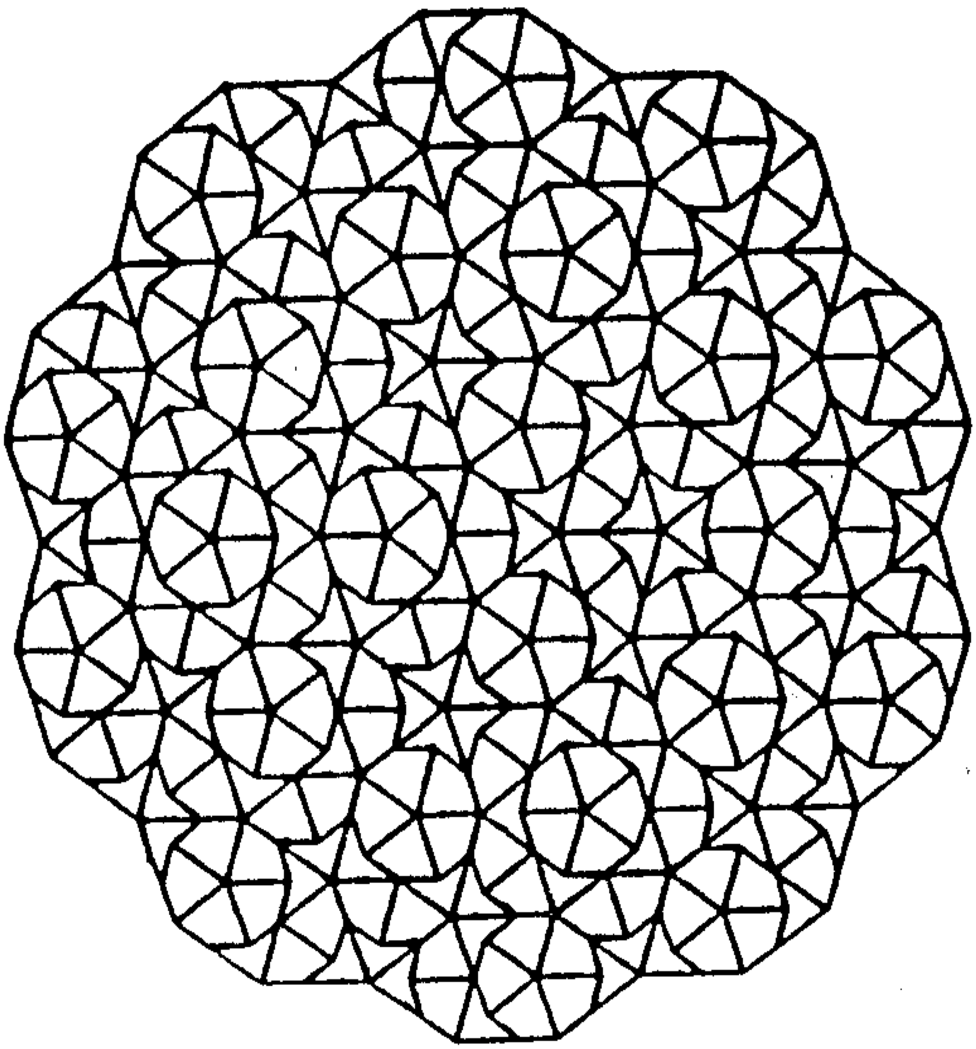

Figure 15
An aperiodic tiling of the plane
Also called a Penrose tiling
As shown in Grunbaum & Shepherd 5 (1986, p. 560)

4.6 Games, puzzles, number theory.

Numbers and their properties have fascinated people in all cultures in one way or another. Some of this fascination stems from the innate preference for play that all human beings seem to possess to some degree. Numbers have the capacity to engage us in play, be it just figuring out patterns, wrestling with puzzles, mystifying our (sometimes unwilling) audience with "magic" tricks based on some properties of numbers, or what have you. Play with numbers is also often intellectually rewarding. Indeed, many mathematicians would agree that mathematical research is often a logical extension of this type of play. The difference lies merely in the level of complexity of the tools and the degree of patience and perseverance needed for the activity. Play with numbers can take many forms: games in which simple properties of division might be involved (such as the game of "20," a member of the family of games known as Nim[62]), puzzles that might require some mental computation[63], or magic tricks that depend on some intrinsic property of numbers. Puzzles or tricks based on properties of numbers lead us naturally into asking why the particular puzzle or trick "works" the way it does and this is just a step away from number theory, the study of the properties of numbers as a subject in its own right. Historically, number theory has been a source of great inspiration for the development of many different parts of modern mathematics. It is a subject full of surprises, with many examples of questions that are easily formulated, but not always as easily answered.

An example of this kind of activity is the design of magic squares. A square array ($n \times n$ say) of numbers is said to be a magic square if each row, each column and the two diagonals have the same number as their sum. Many such magic squares are well-known, and the activity of constructing magic squares dates back to antiquity and

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62 20 is played as follows: A pot contains 20 beans. Two players take turns at making moves. Each move consists of taking either one or two beans out of the pot. The player who is forced to empty the pot will lose.

63 Or sometimes have a trick beginning that leads you into a lengthy calculation that in the end turns out to have been superfluous: for example as in "As I was going to St. Ives, I met a man with seven wives;.each wife has seven sacks, each sack has seven cats, each cat has seven kits. Kits, cats, sacks, wives, How many were going to St. Ives?" or in "You are driving a bus that starts with 20 passengers. At the first stop, 4 get on and 3 get off; at the second stop 2 get on and 4 get off; at the third stop...., at the fourth stop....., ... , at the tenth stop 5 get off and 7 get on. What is the name of the driver?" The St. Ives problem is very old, and goes back to Egyptian times, it appears. It was not then a trick problem, but evolved into one. See Eves (1983, p. 35).

continues to delight people to this day.[64] An example possibly dating back to ca. 1000 B.C. is cited by Stapleton (1953; p. 36),[65] who tells us that it is found "as the ground plan of the *Ming-Tang* – the Ducal (and, later, Imperial) Temple of Mystic Enlightenment." Stapleton (1953; p. 37) also asserts that this "Magic Square was known in Europe to Theodorus, a pupil of ... Porphyry." The example cited is the magic square:

6	1	8
7	5	3
2	9	4

Essentially the same example is discussed by Needham as the *Lo Shu* diagram, ca. 200 B.C., which is shown in Figure 16. Another example is contained in Zaslavsky (1973) who describes the work of an 18th century West African mathematician Muhammad ibn Muhammad. In Needham (1959, p.60) one finds an example of a "magic cube" (i.e., a *three* dimensional analogue of the magic square).[66]

Starting from such antique beginnings, the construction of magic squares with other properties has engaged mathematicians over the ages. The properties can get quite involved. An interesting example is provided by the following magic square that occurs in the famous engraving *Melancholia* by Albrecht Durer:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

64 Rouse-Ball & Coxeter (1939) give many examples and a procedure to construct magic squares of any size.

65 I am indebted to Professor S. Nomanul Haq for supplying me with this reference, quoted in Haq (1994).

66 Professor Li Di comments that the example (of the magic cube) quoted by Needham is from Bao Qui Shou (early Qing dynasty).

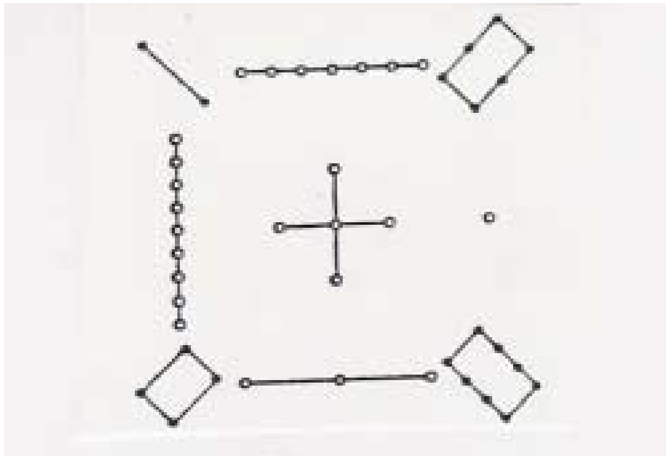

Figure 16
The Lo Shu diagram
As shown by Needham (1959, p.57)[67]

In this square, not only do the rows, columns and diagonals add up to the same number (34), but one also finds the following: (a) The sum of the squares of the numbers in the top two rows is equal to the sum of the squares in the bottom two rows (b) The sum of the squares of the numbers in the first and third rows is equal to the sum of the squares of the numbers in the second and fourth row. (c) The sum of the numbers in the diagonals is equal to the sum of the numbers *not* in the diagonals. (d) The sum of the squares of the numbers in the diagonals is equal to the sum of the squares of the numbers *not* in the diagonals. (e) The sum of the cubes of the numbers in the diagonals is equal to the sum of the cubes of the numbers *not* in the diagonals.

Durer most probably got this square from some person (unknown) who was passionately interested in magic squares. One can easily pose the problem of constructing magic squares with additional properties such as (a), (b), etc. above.

67 The picture represents the square

2	7	6
9	5	1
4	3	8

Note that this square is essentially the same as the square cited by Stapleton, except that the rows of one are the columns of the other (just look at the *Lo Shu* square sideways; you get the *Ming-Tang* square!).

(Such a problem belongs to a class of problems in number theory called Diophantine problems, so called because Diophantus of Greece posed and studied many such problems. In such problems, one seeks solutions to equations in one or more unknowns, with the restriction that the solutions should be non-negative integers). Although easily posed, it is certainly not so easy to solve such a problem.[68] The problem of constructing magic squares is an example of a simple sort of Diophantine problem. Thus, one sees here a thread from an old Chinese problem to a subject that continues to interest mathematicians today.

As we saw, there is a connection between magic squares and problems of number theory that require solutions in integers. Such Diophantine problems occur also in other contexts. van der Waerden (1983, pp. 121-131) has an extensive discussion of the way in which linear Diophantine equations were used in astronomy by the Indian mathematicians Aryabhata and Brahmagupta, already referred to above. Later Indian mathematicians continued the study of Diophantine equations, and in the works of Jayadeva and Bhaskara II there is a treatment of what is now called Pell's equation, which is:

$$x^2 = dy^2 + 1$$

or, in more general form:

$$x^2 = dy^2 + f$$

where d is a positive integer and f is an arbitrary integer, positive or negative. The problem is to find integral solutions for x and y.[69]

This equation was studied in its general form, with an arbitrary integer f (positive or negative) instead of 1 on the right side, by Jayadeva and Bhaskara II (both ca. 1150

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68 The most celebrated example of an easily formulated but very difficult problem is Fermat's last problem: given an integer $n > 2$, find integers x, y, z such that $x^n + y^n = z^n$.

69 It can be shown that if we can find solution with x and y integral, then we can use the solution to get good approximations to $\sqrt{d}$ by means of rational numbers. This was the original motivation for studying this equation. Thus, Pell's equation in its first form but with $d = 2$ or $d = 3$ was studied by Greek mathematicians in antiquity. Pythagoras was interested in it in order to find useful approximations to $\sqrt{2}$. Similarly, Archimedes used it to approximate $\sqrt{3}$. The name "Pell's equation" was invented by Euler in the 18th century. According to Dickson (1919-23, Vol. 2, p. 341), Pell had very little to do with the equation that bears his name.

A.D.). This work of Jayadeva and Bhaskara was not known in the west until about forty years ago. Independently, European mathematicians also studied Pell's equation, and Euler and Lagrange (among others) had written on the subject. Following a suggestion made by Euler, C. O. Selenius developed a method for solving the general form of Pell's equation in Selenius (1962), which he later on found to be equivalent to the method that had been used by Jayadeva and Bhaskara II. See the discussion in van der Waerden (1983, p. 149) concerning this point. This example, although isolated, tells us something about the process of mathematical discovery and the manner in which credit for mathematical discoveries survives in the historical record.

Another problem that occurs in early Chinese work is the so-called Chinese Remainder Problem. It is of great interest because of its importance in modern algebraic number theory. (This branch of number theory, which came into prominence in the 19th century, aims to study properties of numbers by means of the methods of abstract algebra). An important tool in that theory is a theorem called the Chinese Remainder Theorem, which is a direct descendant of the Remainder Problem that was treated in ancient China.[70] It occurs in Sun Tzu's *Sun Tzu Suan Ching (Master Sun's Arithmetical Manual).* According to Needham (1959), this work was written between 280 A.D. and 473 A.D. In it one finds the following problem:

> We have a number of things, but do not know exactly how many. If we count them by threes, we have two left over. If we count them by fives, we have three left over. If we count them by sevens, we have two left over. How many things are there? (See Mikami, 1974.)

Sun Tzu gave in his treatise not only a solution of this specific problem, but also describes how one may solve the more general problem of finding an unknown number from the information given in terms of the remainders that are respectively left over when the unknown number is divided by each integer in a given finite set of integers. The method of solution is recognizably similar to the algorithm that results from the modern proof of the Chinese Remainder Theorem. (see Mikami, 1974.) The method is given as a recipe and cannot be construed as a *proof* showing that the method is correct. However, it is clear that the method must have been tried computationally in

70 The Chinese Remainder Theorem is discussed in most modern books on abstract algebra.

several different cases, in order for the writer to be certain that it worked.[71] The same problem occurs in Brahmagupta's work, in substantially similar form. It was later systematically treated by Bhaskara II by means of the technique known as "the pulverizer."

I end this section with an example of a legendary puzzle, known in many old Asian cultures, that is often used as an illustration of a mathematical method known today as recursion. This is the problem known as the "Tower of Hanoi" problem. See Figure 17.

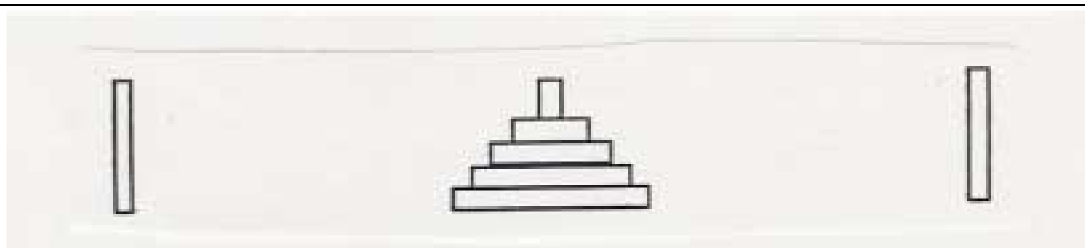

Figure 17
The Towers of Hanoi
An abridged version with just four discs

This problem is based on the legend that in a remote temple in the ancient kingdom of the Khmer people (its capital city, Angkor, is in present day Cambodia) certain priests were set the following task by Brahma, the Creator, at the instant of creation of the universe. In the courtyard of this temple were three pegs. On the first of these pegs, there was a pile of 64 stone discs, each with a hole in the middle. The discs were graded in size and were arranged pyramidally, i.e., in decreasing order with respect to size, the largest disc being at the bottom, the smallest one being on top. The priests' task was to transfer the pile to one of the other two pegs, moving one disc at a

71 Professor Li Di informs me that Sun Tzu's treatment of this problem was restricted to a special case, namely when the finite set of divisors are mutually relatively prime. The general case of *any* finite set of divisors was treated by Chhin Chiu-Shao, in his work *Shu Shu Chiu Chang Zhang* (ca. 1247 A.D.). A reference to this latter work can be found in Needham (1959). This problem was later treated by C.F. Gauss in 1801.

time from one peg to another, *with the proviso that a larger disc must never ride atop a smaller one in the process of transfer*. The legend continues thus: the time taken for this task is identical to the lifetime of the universe; the instant that the task is finished the world will come to an end! (Thus, in some remote corner of the South East Asian jungle, those priests must be busily plying the discs).

One can ask the question: How many moves will be needed to affect the transfer of the discs from one peg to another? This is a problem that is today quite often used in elementary college courses as an illustration of the technique of *recursion*.[72] (In the context of this problem, the technique of recursion, roughly speaking, refers to the fact that one finds a relation between the given problem and a *smaller* version of it, which relation is then used *repeatedly* to reduce the problem to the smallest version imaginable, which is easily solved).

Using that technique, it can be shown fairly easily (see footnote 72) that the number of moves needed for the original problem with 64 discs is equal to $2^{64} - 1$. Now $2^{64} - 1$ is a very large number and it is not surprising the legend equates it to the lifetime of the universe. In fact, if the priests took just one second to move each disc, one can estimate easily that $2^{64} - 1$ seconds is well *over thirty trillion years*, longer than the

72 For the sake of the reader who is curious about the method by which the Tower of Hanoi problem is solved, I give here the proof for the case of 4 discs: The number of moves that is needed to affect the transfer clearly depends on the size of pile. Suppose that the number of moves needed for the pile of size 4 is M(4). Assume that we have figured out how to do it for the case of 3 discs, and that the number of moves needed is M(3). We shall show that M(4) = 2M(3) + 1, by the following three steps. This provides a connection between the problems of size 4 and size 3.

Step 1: Starting with the pile of size four, we decide to leave the bottom disc undisturbed, and operate with the top three discs, and transfer them to one of the other pegs, say the one at the left, taking M(3) moves. (Notice that we can use all three pegs for doing this, because the bottom disc on the center peg is the largest of all, so any one of the other discs can be placed over it, without violating the ground rules.) At this point we have the top three discs arranged on the leftmost peg, and the largest disc is still on the center peg.

Step 2: Now move that disc to the peg on the right. This takes one move.

Step 3: Now move the three discs from the leftmost peg to the rightmost one, again taking M(3) moves to do it.

The task is finished with these three steps. The total number of moves is M(3) for step 1, 1 for step 2, and M(3) for step 3, equaling 2M(3) + 1 steps. Therefore, this is the number M(4) of steps needed to accomplish the transfer of 4 discs. A similar argument shows that M(3) = 2M(2) + 1, and M(2) = 2M(1) + 1. But M(1) is easily calculated. It is the number of moves needed to transfer a pile of size 1. Clearly we can do this in one move, so we know that M(1) = 1, which is $2^1 - 1$. Working backwards, we find that $M(2) = 2M(1) + 1 = 2(2^1 - 1) = 2^2 - 1$, $M(3) = 2M(2) + 1 = 2(2^2 - 1) + 1 = 2^3 - 1$, $M(4) = 2M(3) + 1 = 2(2^3 - 1) + 1 = 2^4 - 1$. The same procedure can be used to show that for a pile of size *n,* we have $M(n) = 2^n - 1$

estimated lifetime of our universe to this day![73] The legend is known in different versions in different parts of Asia. For example, here is an Indian legend (which is similar to a Chinese legend): An arrogant king had a learned man at his court. Once this learned man gave the king some advice which proved very useful. The king said to the learned man that he could ask for any gift. The learned man did not lack for anything, but wanting to teach the king a lesson in humility, he said, "O King, I only need a little rice. See this chessboard here. Give me one grain for the first square of the chessboard, two grains for the next square, double that for the next square, and so on, until all squares are accounted for. That is all I ask." The king arrogantly saying that this is a trivial wish, ordered the keeper of the royal store to attend to this trivial amount of grain, only to find that long before the chessboard could be even partially covered, his granary had run into bankruptcy.[74] All these cultures seem to have been well aware that the number of moves needed for the solution is extremely large. There is no evidence, of course, that they had a fully understood theoretical basis for coming to this conclusion, but there clearly was an empirical understanding of the phenomenon of geometric growth that underlies these legends.

In the confines of this essay, I cannot discuss other examples of the type of inspiration provided by games and puzzles for the development of different areas of mathematics. Although limited, I hope that the examples that I have dealt with will give the reader an idea of the scope and variety of such inspirations. Rouse-Ball & Coxeter (1939), Dudeney (1967) and especially Gardner (1961, 1972, 1987, 1989, 1995), are excellent sources for further information on this point.

73 So we should not be surprised if those priests are still at it in some remote part of the jungle, hitherto undiscovered by prying eyes.

74 No wonder. The number of grains needed is $1 + 2 + 2^2 + 2^3 + ... + 2^{63} = 2^{64} - 1$. This is clearly a version of the same legend!

5. Mathematics in Asia after the 16th century.

The preceding section contains examples of many mathematical questions that interested people in certain Asian culture-areas at various times in history. The time period they cover is quite long, starting from several hundred years B.C. to about the 15th century A.D. Although the historical record is certainly not complete, it is coherent enough to give us a good picture of the concerns and interests of the cultures that pursued these mathematical questions and the depth and sophistication of their researches. It is natural to try and understand how the contributions of these culture-areas relate to mathematics of our time — i.e., to seek to form a picture of mathematical developments in Asia from the 16th century onwards, to our own time.

The picture that emerges can be summarized as follows.[75] At the middle of the 16th century, the level of mathematical sophistication that had been reached by Asian investigators was fully comparable to mathematics in Europe. In the 150 year period consisting of the second half of the 16th century and the entire 17th century, an astonishing sequence of advances were made by European mathematicians that fundamentally changed the direction of mathematics.[76] In earlier years, mathematics was concerned largely with problems of calculation and measurement, as they would arise in the occupations of what Needham has called the "higher artisanate" — i.e., surveyors, metallurgists and weapon makers, calendar makers, navigators and so on. Starting from the latter part of the 16th and 17th centuries, European mathematics began to make deeper connections with the natural sciences by seeking theoretical explanations of natural phenomena (a development that can be traced to the point of view typified by Galileo). At the same time, it began an inexorable march towards greater precision and abstraction, broadening its scope from the study of number and shape, until in time it could be described as "the science of structure and pattern" (in Lynn Arthur Steen's felicitous words). The mathematics of our time can be seen to be a

75 I am aware that this is a vast subject and that whatever one says in a few pages is bound to sound oversimplified. For an extensive discussion of the issues of this section as they relate to China, see Needham (1959, pp. 152-167). Similar comments apply to India and Western Asia as well. See also Huff (1993). Another good popular account is Lindberg (1992).

76 Thus we have, e.g., Algebraic notation, 1580 (Vieta and Recorde); a proper explanation of decimals as fractions, 1595 (Stevin); logarithms, 1614 (Napier); slide rule, 1620 (Gunter); analytic geometry, 1637 (Descartes); an adding machine, 1642 (Pascal); the differential calculus and the beginnings of the integral calculus, 1665 (Newton) and 1684 (Leibnitz).

natural outgrowth of this fundamental change in the internal dynamic of mathematics, enriched by 18th and 19th century developments.

For a number of different reasons, which will be alluded to later, this qualitative change substantially bypassed all the culture-areas of Asia that I have discussed — China, India, West Asia.[77] The result was that the mathematical work done in those areas was, after the 16th century, out of tune with the paradigms that inspired the great discoveries of mathematics in the 18th and 19th centuries. Understanding the relation between the emergence of circumstances of the time is part of what Needham (1959, p.167) calls "the Great Debate of the History of Science in Europe." Understanding the cultural and social reasons that might have inhibited the emergence of a similar point of view in Asian culture-areas is a complementary question that is equally important. I shall briefly consider this question.

In any society, healthy development of mathematics (and more generally of science or any other intellectual endeavor) depends on a variety of factors. Among the factors that are conducive to a milieu of vigorous mathematical inquiry are: political and social stability, a communal mathematical culture in which interaction is frequent and easy, adequate support of mathematics by society, and adequate social status for the pursuit of mathematics. (Indeed, an analogous statement applies with equal force to all fields of inquiry.) An examination of the factors that might have inhibited a continuation of the mathematical tradition of the previous years in these societies is beyond the scope of this essay and in any case far beyond my competence. However, looking cursorily at the history of these regions, one arrives at some understanding of a few of the forces at work, which clearly had an impact on the development of mathematics and science in these areas. These are: (a) political instability, (b) the impact of the era of European colonial power, (c) the structure of patronage and support of mathematics and science, and (d) cultural and social factors that influenced the directions of inquiry.

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77 Although, as Needham (1959, p. 160) notes, one of the big ideas that drove European science, namely the idea that the objective universe needs to be studied and understood for its own sake, was rooted in the work of the scientists of the ‘Abbasid period. European science took it to a transendant level.

5.1 Political instability.

Political instability was certainly a factor in all three regions. In China, the Ming dynasty (1368-1644), during which political and social conditions in China were relatively stable, was followed by the Qing dynasty (1616-1911). The early parts of this period saw the central regime dealing with Manchu insurrections and the latter part was already fully in the colonial era, during which the infamous opium wars were fought.[78] Similar instabilities beset the Indian subcontinent after the 13th century.[79] Relatively stable conditions prevailed after the advent of the Mughal empire, especially during the reign of the emperor Akbar (1556-1605), but the stability was already being challenged by the middle of the 17th century by the Marathas, and although power was nominally stable, the empire had to be continually defended at its boundaries. Conditions in South India were not more stable either. The 17th century also saw the growth of colonial power in India, which had become paramount on the subcontinent by the middle of the eighteenth century. In West Asia, the instability ushered in by the sack of Baghdad by the Mongols in the middle of the 13th century marked the beginning of a period of political instability that accelerated after the fall of the Western Caliphate in Cordoba, in 1492. With the development of ocean routes for the trade with Eastern Asia in the next few decades, the importance of the Silk Route diminished rapidly and so did the prosperity and power of the Caliphs.

5.2 The impact of colonial power.

Added to all these internal political and cultural factors that impeded the continued development of mathematics, there was also the external factor of colonial power, which affected Asia profoundly in a variety of ways. First of all, there was the economic impact of colonial power. In countries that were directly under colonial rule,

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78 In the intervening 100 years or so, there was a period of tranquility (which some have called the *pax Sinica*), during which many advances in Chinese society took place. However, there were also serious weaknesses in the social fabric, corruption and nepotism being widespread, and the rule of the bureaucracy being nearly absolute and ruthlessly wielded. See Needham (1959).

79 North India had always been a feudal society, very unstable, and quite fragmented politically throughout the 14th and the 15th centuries, even when formal sovereignty was enjoyed by some kings over a large parts of the country. Pockets of culture that had permitted mathematical activity to bloom during earlier times could not survive these harsher times; of course, the general unrest was probably inimical to the emergence of fresh traditions as well.

the impact was obvious. Even in countries that were not formally under a foreign government, the economic impact of European colonial power was felt in a number of ways that mostly downgraded the economic and political viability of those countries (e.g., the impact on China of the Opium wars). In countries in which colonial rule was manifested by direct foreign government (e.g., India), there was, in addition, the stultifying psychological impact arising from the implicit assumption of superiority which colonial rulers found it natural to make. Even in its more benign aspect, colonial rule was relentlessly patronizing. This had a practical effect on the rebirth of indigenous science in colonial countries after western education had been introduced there by the rulers. It meant, for example, that the products of the indigenous system could never hope to be rewarded for their intellectual achievements unless they also acquired the imprimatur of an education in the ruling country. (In India, for example, even though a nominally extensive system of university education had been introduced by the British, the implicit assumption was always that the system could never hope to attain the status and the sophistication of the British universities). It is clear that this circumstance alone would be fatal to the growth of a healthy infrastructure for indigenous science.

5.3 Patronage and support of mathematics and science.

The support of mathematics and science in many historical cultures came about through the patronage of rulers. In the case of the three culture-areas that have been the focus of this essay, this is particularly clear. The enlightened attitude of three successive Caliphs (especially al-Ma’mun, who was an active scholar himself) towards science and mathematics was a crucial ingredient in the development of science in the ‘Abbasid period. Similarly, the encouragement given to scholars in India during the Gupta period was important for the development of science and mathematics. In China, there had been a long tradition of royal support for scholarship, which played a critical role in the history of mathematics in China. (see Needham, 1959.)

Royal patronage is dependent on the whim of the ruler however, and during times in which the interest taken by the ruler in a particular field waned, the support for scholarship in that field was likewise bound to wane. Thus, for example, in India the focus of royal patronage seems to have shifted away from science and mathematics towards other fields after the 16th century (except for one or two notable exceptions, such as at the court of the astronomer-prince Raja Jai Singh in the 17th century). Similarly in China, in the late Ming period, a shift in patronage away from science and mathematics seems to have begun.

5.4 Cultural and social factors.

Mathematics, like all other human activity, has a cultural and social context that influences the choice of questions that mathematicians single out for study, as well as the depth to which such a study is carried. In understanding the reasons why the course of mathematics in Asia after the 16th century ran orthogonal to the inspirations that were driving European mathematics, perhaps one needs to consider the influence of cultural and social factors, above all others. There is a wealth of written material and a variety of points of view on issues surrounding this question. It may be useful to summarize briefly some recurring themes.

1. Certain key mathematical ideas were not adopted or internalized. For example, Mikami (1913) points out that the idea of a formal proof never became a part of the Chinese tradition, thereby inhibiting the development of a formal logical exposition. Similarly, mathematicians in India did not devote much attention to geometry, in comparison with the attention they gave to number theory and algebra, in spite of considerable exposure to these ideas via their contacts to West Asia.

2. Social status accorded to those engaged in science was a key factor. In China, for example, mathematicians were for the most part "plain practical men, men in the retinues of provincial officials" (Needham, 1959, p. 150ff). They were employed mostly for bureaucratic calculations dealing with land taxation and the like and were valued for those services. High status in society came about via the civil service, in which accomplishment in other areas, such as calligraphy and poetry, were prized. In India, mathematics was for the most part regarded as a tool, mainly for astronomical calculations motivated by astrological application. After a fairly accurate system of calculating planetary positions had been perfected, there was no impetus to accord mathematicians recognition for the sake of their basic mathematical researches, which were regarded as esoteric by most people (Datta & Singh, 1935). In West Asia, the decline of the Caliphs was followed by a period during which court patronage for

mathematics was absent and tastes shifted towards religious and philosophical discourse.

3. Philosophical views of the world that prevailed at the time had an inhibiting effect on inquiry into the laws of nature. To quote Needham, "Lastly, a factor of great importance must be sought in the Chinese attitude to the Laws of Nature." Again, according to Needham (1959, p.153), "The firm conviction (expressed by Taoist philosophers in high poetry of Lucretian vigour) that the whole universe was an organic self-sufficient system, led to a concept of an all-embracing Order in which there was no room for the Laws of Nature, and hence few regularities to which it would be profitable to apply mathematics in the mundane sphere." Similar comments about the preoccupation with philosophy and textual studies in India and West Asia have been made by other scholars in support of this point.

4. The rise of mercantile capitalism (and, subsequently, industrial manufacture) together with the formation of cities in Europe provided powerful economic incentives and also created an institutional infrastructure (via guilds, universities, etc.) for scientific scholarship. The absence of a similar parallel development in Asia inhibited the growth of science and mathematics in a major way. Moreover, as the center of gravity of economic prosperity shifted to Europe, partly due to efficiencies of the Industrial Revolution and partly due to the expansion of European power by colonization, it became harder for these culture-areas to carry out the investment needed to build that infrastructure.

5. The social forces mentioned in the previous paragraph also legitimized and accorded higher status to pursuits of scientists and mathematicians. As Needham (1959, p.155) puts it: "Part of the story undoubtedly concerns the social changes in Europe which made the association of gentlemen with technicians respectable" (not to mention profitable). Books, both theoretical and practical, began to appear and were quickly disseminated in the growing web of urban communities linked together with better transportation. In contrast, in China, "Books were all essentially 'traditional', the product of slow growth under bureaucratic oppression or at best tutelage, not the creations of enterprising merchant ventures with big profits in sight." Needham (1959, p.167).

6. One very important factor in the development of European science was the formation of a pan-European community of mathematicians and scientists. Discourse between its members led to important advances. The advent of printing and the creation of the infrastructure mentioned above facilitated and nourished this community.[80] In the years during which mathematics flourished in the culture-areas that were considered above, one can detect a beginnings of such a community, especially at the ‘Abbasid court. However, for a number of reasons, such a community did not survive the political maelstroms that befell those areas and was never revived.

7. Many scholars feel that perhaps the most important factor was that the Galilean point of view (which regarded mathematics as the natural language with which to describe the *causal structure* of natural phenomena, rather than merely a tool to calculate their impact on daily life) was never fully accepted by scholars in these three culture-areas. (See the extensive discussion in Needham {1959, pp. 150-168}, where many other references are given). This change in the point of view elevated mathematics to a new level of inquiry. Needham gives a long list of Chinese achievements in fields such as metallurgy, architecture/engineering, gunnery, and so on, which were fully comparable to the best works of similarly placed and famous members of the "higher artisanate" in the Renaissance (e.g., Leonardo da Vinci). He goes on to note, "But in Europe, unlike China, there was some influence at work for which this stage was not enough. Something pushed forward beyond it to make the junction between practical knowledge, empirical even when quantitatively expressed, and mathematical formulations." He goes on to note that this connection between the natural sciences and mathematics was, in his view, the most important factor in the development of European mathematics. The lack of a similar development in China was a major reason for the relative decline of science and mathematics in China.

80 That such a community is the decisive mechanism for the transmission of mathematical knowledge has been persuasively argued by Kitcher (1983). Kitcher goes much further, arguing against the commonly held philosophical view that mathematical knowledge is *a priori* – i.e., it is obtained from a source other than perceptual experience, and hence fundamentally different from knowledge in other natural sciences. Kitcher's position, roughly speaking, is that mathematical knowledge follows an evolutionary pattern; the knowledge of individuals being derived from community authorities, and the knowledge of a community being grounded in the knowledge of previous communities.

Similar observations can be made for India and West Asia as well. I end this discussion with the following quote that summarizes the above points very well:

> (In Chinese science) there came no vivifying demand from the side of natural science. Interest in Nature was not enough, controlled experimentation was not enough, empirical induction was not enough, eclipse-prediction and calendar-calculation were not enough; all these the Chinese had. Apparently a mercantile culture alone was able to do what agrarian bureaucratic civilization could not - bring to fusion point the formerly separated disciplines of mathematics and nature-knowledge (Needham, 1959, p.168).

Scholars seem to agree that the Galilean revolution was the temporal landmark after which the paths of European and Asian mathematics (and science) began to diverge. Although one cannot say with certainty what factors ensured the spectacular development of European science, one may nevertheless make a couple of reasonable inferences from what one knows of the history of that development, as sketched above. One is that healthy institutions that can support and nurture free inquiry are indispensable to the success of the scientific enterprise in any culture. Another is that a healthy dialogue between various sciences and mathematics has been a source of progress in both and has contributed in a large measure to the economic prosperity of Europe (and the west in general). The third is that the continuance of a healthy scientific enterprise cannot be taken for granted; an effort must be made to understand the factors that promote it and policy must be reformulated so that those factors are institutionalized, if possible. As the United States goes into the 21st century facing not only unprecedented political challenges globally, but also facing (again for the first time in her history) the specter of a possible decline in her global position as the most prosperous nation in the world, it would be well for the citizenry to be aware of these issues. A healthy scientific enterprise is surely one of the key ingredients necessary for the continued future prosperity of the United States.

Although the history of Asian mathematics in the period from the 16th to the 19th century is pale in comparison to earlier periods, developments during the last part of the 19th century and in the present century have enabled many Asian countries to re-establish their mathematical enterprise along lines that are promising. Thus, the last

150 years have seen the establishment of universities in essentially all countries in Asia, leading to the prospect of scholars in all those countries participating eventually in a global community of scholars. The colonial era is over politically (although its economic effects survive, and are far from negligible).[81] Many countries in Asia today are on a healthy track of improving economic conditions and are acutely aware of the role played in the sustenance of that trend by a strong science and mathematics education. In turn, this has led to greater interaction between mathematicians in those countries and western countries in which a strong tradition of mathematical research is well-established. Although the extent of such interaction varies greatly from one country to another due to differences in their specific situations, one can legitimately claim that, compared to the situation that prevailed 100 years ago, giant strides have been made towards the establishment of a global community of mathematicians. Nevertheless, much remains to be done in this direction.

In this process of increasing interaction with and impact on the global mathematical research, China, India and Japan have had the most visible set of interactions in the 20th century. In Japan, for instance, after the Meiji restoration and the subsequent assiduous cultivation of contacts with Europe, especially Germany, a university system of high quality was established and many mathematicians of very high ability have emerged from that system. Among literally scores of Japanese mathematicians who have made important contributions to 20th century mathematics, I mention just a few (listed in alphabetical order of last names): Goro Azumaya, Heisuke Hironaka, Yasutaka Ihara, Kiyosi Ito, Shizuo Kakutani, Kunihiko Kodaira, T. Morita, Tadasi Nakayama, H. Okamoto, Goro Shimura, Ishiro Takagi, M. Takesaki, and Y. Taniyama. Similarly, there have been very fine contributions by mathematicians from India. I mention just a few names: Laxmi Bai, Harish-Chandra, S. Chandrasekhar, K. Chandrasekharan, M.S. Narasimhan, R. Narasimhan, Gopal Prasad, M.S. Raghunathan, K.G. Ramanathan, Srinivasa Ramanujan, C.R. Rao, C.S. Seshadri, Bhama Srinivasan, V.S. Varadarajan, and T. Vijayaraghavan. Among mathematicians of Chinese heritage, I can mention: Alice Chang, S.S. Chern, W.L. Chow, Fan Chung, W.-C. Hsiang, W.-Y. Hsiang, L.K. Hua, P. Li, C.C. Lin, Y.T. Siu, S.T. Yau, and H.

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81 Of course, inequalities of wealth (resulting in woefully inadequate libraries and laboratories, for example) often make it hard for them to participate on an equal footing. Indeed, the inertia of the inequalities of wealth caused by the history of the colonial period may be significant enough to thwart the full participation by sizable numbers of scholars from many Asian nations in an international community. See, for example, Thurow (1992) for a global perspective on this general issue.

Wang. The contributions of these mathematicians span all the major subfields of modern mathematics: algebra, number theory, algebraic geometry and analysis, etc.[82]

82 These lists are partial, and can be easily extended. They are in no sense lists of "The N Best Mathematicians." In any event, I ask the reader to bear in mind that such lists will obviously suffer from my limitations.

6. Mathematics in the United States; contributions of Asian-Americans.

6.1 Mathematics in the United States.

In dealing with mathematics in the United States, I shall restrict myself to the post-colonial period. The written record of mathematical practices of American Indians is very sparse, and of a character that does not connect readily to the issues discussed in this article. Moreover, I am not qualified to deal with this aspect. The interested reader may wish to consult Closs (1986) or Landon (1993) and the references cited there for further information on this point.

I have remarked above on the astonishing development of mathematics in post-Renaissance Europe. A convenient temporal landmark is the era of Newton and Leibnitz, from the middle of the 17th century to the end, more or less. As a result of their work, the calculus emerged as the tool that could be used to understand a very large part of the mathematics of that time. No wonder then that a vigorous development of related branches of mathematics soon followed the discovery of the calculus (and shortly thereafter, the discovery of analytic geometry). The next two centuries in Europe saw the development of a mathematical culture of great vitality, most of which survives to our day. Simultaneously, one sees the beginnings of the Industrial Revolution in Europe, accelerating the pace of urbanization that had already started earlier in Europe as it emerged from the old agrarian-feudal order into the era of commerce and civic organization. The interaction between mathematics on the one hand and the sciences, engineering and technology on the other, proved to be symbiotic interaction for both. These centuries spanned almost precisely the colonial period in North America. Institutions of all kinds were in their pre-natal or infant stage, and one cannot discern (but for isolated exceptions) any significant contributions to mathematics from North American mathematicians during this period.

Even after the emergence of the U.S. Republic, one does not see anything like a U.S. tradition of science or mathematics emerging until towards the end of the 19th century. In contrast, European science was undergoing a vigorous development. After the exuberant expansion of the calculus and its applications in the 17th and early 18th centuries, there was now a systematic investigation into algebra, number theory, and geometry. During the 18th and 19th centuries, European science developed at a rapid pace and produced an impressive number of contributors in all branches of science and mathematics. The framework of research directions in mathematics that was laid out by the mathematicians of that period is influencing the direction of mathematical work even today. While it is not possible for us to describe their work in an elementary manner, it is easy to appreciate the atmosphere of tremendous intellectual ferment that characterized this period.

Although the United States had the beginnings of the kind of institutional structure that is needed to support scientific research quite early (witness the early founding of a number of our eastern universities), the general character of United States science and mathematics was fairly naive until the latter part of the 19th century. The few mathematicians who contributed anything of significance were mostly European scholars who served for a few years at U.S. universities, which acted as way stations for them in their careers. It is only towards the end of the 19th century that the beginning of an indigenous tradition in U.S. science and mathematics can be seen. By this time (i.e., in the last twenty years of the last century and the first few years of our own) there were a number of U.S. universities that were striving to attain the same standards in the United States that the leading members of their faculty had experienced in Europe (A significant proportion of the intellectual leaders in their faculties had been trained in European universities). A generation of U.S. born scientists and mathematicians, having studied in Europe, now began to trek back to their home institutions, to try to put U.S. science and mathematics on a par with the rest of the world. Between 1900 and 1930, their influence began to have an effect, and U.S. science and mathematics could be referred to as recognizable entities. A number of other factors also fostered the emergence of the United States as a significant force in global scientific life: the rapid development of technology in the late 19th and early 20th centuries, resulting in a stream of inventions (e.g., the automobile, the radio, and the telephone to mention just three); the end of an isolationist phase in U.S. foreign affairs (e.g., signaled by U.S. participation in the first world war); and high and sustained rates of economic growth. All these factors played a role in this process of emergence.

The growth and the success of U.S. science and mathematics after 1930 has been simply phenomenal. The U.S. role in the second world war (including the development of the atomic bomb) and during the structuring of the peace that followed established the United States as a superpower and forced it to play a role as a leader of the free world from which there was no possible reprieve. Nor were U.S. institutions found wanting in the task of supporting scientific research. Just after the war, after getting a top level report[83] about the future of scientific research, President Truman signed into law a bill establishing the National Science Foundation. In the ensuing 40 or 45 years, Federal support of science and mathematics, through the National Science Foundation and other agencies, combined with a rapid expansion of the public university system in the United States and a pattern of steadily increasing levels of prosperity have helped in establishing the United States at the forefront of science and mathematics research in the world.

The role played by immigrants in this development (as in countless other areas of U.S. life) has been significant. Throughout (and especially in the latter half of) the 19th century, migration from Europe was at very high levels. Although these entrants to U.S. society suffered many of the pains of adjusting to a new social order, over the span of a generation they not only adjusted to the new country, but their offspring often proved to be of very vigorous stock, making distinguished contributions to every walk of life. During the 1930s and 1940s, there was a sudden large influx of Jewish mathematicians (and scientists), who were fleeing the brutal Nazi regime in Germany. Their influence on U.S. mathematics cannot be overestimated. These immigrants made a huge difference in elevating the standards and the output of U.S. mathematical research. Their presence hastened the recognition by the rest of the world that U.S. mathematics had come of age.

6.2 Asian-Americans in U.S. Mathematics.

Asian immigration to the United States began on a sizable scale in the latter part of the 19th century. The earliest Asian immigrants were mainly from China and Japan;

83 This is the Vannevar Bush report, named after Vannevar Bush, who was at that time a scientific advisor to the President.

later there were also considerable numbers of immigrants from the Philippines and India. They came to the United States, like almost all immigrants, for diverse economic or political reasons, usually as indentured laborers to work on the farms on the west coast, or in the construction of the railroads. Another sizable wave of immigration took place at the beginning of the 20th century, when a number of Syrian and Lebanese immigrants came to the east coast and eventually settled in a number of communities near Detroit.

The pattern of recent immigration from Asia is closely tied to global political events. The role of the United States as a superpower, leading, as it did, to U.S. involvement in a number of regional conflicts (willingly or otherwise), clearly affected subsequent patterns of immigration. The wars in Korea and Viet Nam, the establishment of Israel and the subsequent displacement of Palestinians, the civil conflict in Lebanon, the decimation of the rural Cambodian population by Pol Pot, the Soviet invasion of Afghanistan, the overthrow of the monarchy in Iran: all these events have affected the pattern and the composition of Asian immigration to the United States. Today, the United States is irrevocably a multiethnic society.

Descendants of the first waves of Asian immigrants are today fully assimilated in every walk of life in America. Asian-Americans have as a group been very assiduous in the pursuit of education and have set a high value on hard work that leads to academic success. Many of them have become successful professionals: doctors, engineers, scientists or educators. On the other hand, newly arriving Asian-American immigrant groups, although not identical in background and culture to earlier arrivals, have nevertheless often shared similar values for education and have realized fully that attaining a high level of education is their passport to success in U.S. society. It is no surprise that several members of both groups have attained prominence in mathematics. The mathematics faculty of almost every university in the country has some Asian-American members. Scientific research laboratories and Government research agencies or laboratories also have a significant representation of Asian-American mathematics professionals. At present, U.S. graduate programs in mathematics have a very significant proportion of foreign students, many of whom are from various parts of Asia, with a strong representation of students from China. These students are performing very well and some of the best mathematical work done by young mathematicians can be ascribed to them. Many Asian-Americans are known for their fundamental contributions to mathematics in this century. They have been included in the list of names given at the end of section 5.[84]

I want to mention here that although singling out a group of mathematicians on the basis of their ethnic origin may be appropriate for the objective of this baseline essay, my experience has been that the current ethos in U.S. mathematics is remarkably free of any consciousness of the ethnicity of the contributor of a piece of research or mathematical writing.[85] Mathematical research and dialogue usually take place at a fairly objective level. Prejudice evinced in debates about mathematical matters is usually an intellectual prejudice, often stemming from habits of thought long held and therefore not easily surrendered. The writer's ethnicity has little bearing on the debate. (Mathematics, a subject in which objectivity about the subject matter is plausible, is fortunate in this respect). One of the most impressive achievements of the U.S. mathematics community has been the degree to which it has been open to high quality entrants irrespective of national origin.[86]

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84 At the frontiers of mathematical research the definition of Asian-American tends to lose some precision. I include in this category persons who are of Asian heritage, whether born in the United States or elsewhere, who are now resident in the U.nited States and active in mathematics. To illustrate the dilemma that can occur, consider a case (not fictitious) of a mathematician who was born in Asia, immigrated later to the United States, taught and worked for 20+ years in the United States made significant contributions to mathematics, and subsequently returned to his native country on a high academic appointment. Does one consider his contributions as Asian-American or not? The U.S. mathematical research community is so open, compared to its counterpart most other nations, that its membership is quite fluid. Of course, this is partly due to the nature of the subject.

85 This is not to deny that in social matters, or matters involving employment or rewards (e.g., promotion) biases can intervene in decision making. This is, I am afraid, a continuing problem for minorities, and not only in the United States.

86 Being a member of such a community for many years in the United States has been a great source of satisfaction for me personally.

7. Appendix: Origin and transmission of mathematical ideas.

In my interactions with teachers and students of mathematics in schools, I often encounter a set of questions that are best described as being in the category: "Who did it first?" or "Who learned it from whom?" Such questions are not confined only to mathematical topics, of course. For example, students often ask: "Did the Hindus invent the zero?" "Did the Persians invent algebra?" "Did the Chinese invent guns?" And so on. Answers are often expected to be unambiguous, and in the discussion, there can be an undercurrent of the question: "Are we smarter than them or were they smarter than us?" and the like. Ignoring the competitive element in these questions, one can see that these questions have a serious basis and are simple versions of questions such as: How is mathematics discovered; i.e., what is the process by which mathematical knowledge grows? How do mathematical ideas get transmitted from one culture-area to another, especially if they are widely separated in space or time?

These are important questions that involve philosophy as well as history, and they have interested a lot of thinkers in every epoch in the development of Mathematics. Although they are clearly not central to the subject of this essay, they have some bearing on attitudes of cultural superiority, and I would like to address them very briefly in this section.[87]

Traditional thinking about mathematics reflects an almost unanimous view that mathematical knowledge is different from knowledge in the natural sciences. One difference is that mathematical knowledge seems to have a level of certitude that can be set as standard for knowledge in other areas. The other difference is that mathematical knowledge does not appear to be based on experiment and hence seems to be arrived at independently of perceptual experience. In other words, mathematical knowledge has an *a priori* basis and intuition often plays a role (which

87 In doing so, I am ignoring the suggestion of one reviewer to omit such discussion and "stick to facts." I have acceded to the opinions of several high school teachers who felt that a short section would be useful.

cannot be fully explained) in the process of mathematical discovery. Many eminent philosophers of mathematics have held such a view, referred to as *mathematical apriorism*.[88]

The popular view of mathematics is similar. Most people feel that mathematics is mysterious; that mathematical facts are discovered by flashes of insight, by the exercise of a gift somewhat like revelation. They also feel that the type of inspiration needed for a new mathematical idea is rare and hard to duplicate.[89]

In recent times, another point of view has found expression. This view, termed *mathematical empiricism*, holds that mathematical statements have an empirical or partially empirical basis. These views recognize the importance of social and cultural factors; for example, the powerful influence of community in providing an individual's knowledge base and in molding that person's ideas about mathematics.[90] This point of view is explained and considerably extended by Kitcher (1983). A quote from that book sums it up well:

> I shall explain the knowledge of individuals by tracing it to the knowledge of their communities. More exactly, I shall, suppose that the knowledge of an individual is grounded in the knowledge of community authorities. The knowledge of later communities is grounded in the knowledge of earlier communities.

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88 For example, Descartes, Kant, Frege, Hilbert, Brouwer, Weyl. In this connection see the introduction in Kitcher (1983, p. 3). Kitcher says in fact that "Most of the disputes in the philosophy of mathematics conducted in our century represent internal differences of opinion among apriorists."

89 An extreme form of this point of view leads to the hypothesis that similar mathematical ideas found in different cultures must have come from a common source predating those cultures. For example, van der Waerden (1983, p. 10) makes the following argument for the hypothesis of a common source. "With very few exceptions, great discoveries in mathematics, physics, and astronomy have been made only once. Epicycles and eccenters, the spherical form of the earth, the heliocentric system, the three laws of Kepler, the three laws of Newton's mechanics, the law of gravitation, they all were discovered only once. The same holds for the laws of optics, of electricity and magnetism, and so on. In mathematics, there are a few cases of independent invention, for instance in the discovery of non-Euclidean geometry by Gauss, Bolyai and Lobatchevski, but the overwhelming majority of great discoveries in geometry, algebra and analysis were made only once. Therefore when one finds that a great and important theorem, like that of Pythagoras, which is by no means easy to find, is known in several countries, *the best thing to do is, to adopt the hypothesis of dependence*" (my italics). The undeniable fact is that mathematics is extremely effective in a wide variety of unexpected applications. This only adds to the sense of mysteriousness. See in this connection Wigner (1960).

90 Again, see Kitcher (1983, p.4). Kitcher cites Quine, Putnam, Popper and Lakatos as examples of philosophers who hold this point of view, and that it can be traced back to John Stuart Mill.

> At the most recent end of the chain stand the authorities of our present community - the teachers and textbooks of today. Behind them is a sequence of earlier authorities. However, if this explanation is to be ultimately satisfactory, we must understand how the chain of knowers is itself initiated. Here I appeal to ordinary perception. Several millennia ago, our ancestors, probably somewhere in Mesopotamia, set the enterprise in motion by learning through practical experience some elementary truths of arithmetic and geometry. From these humble beginnings mathematics has flowered into the impressive body of knowledge which we have been fortunate to inherit (Kitcher, 1983, p. 5).

I believe that most working mathematicians would take a view somewhere in between these two poles. A typical view is that new mathematical ideas are given form when, following a period of intense contemplation, someone succeeds in describing coherently the essential features of a mathematically interesting problem and perhaps in proposing a method of attacking it. (The problem may or may not have an empirical basis). This formulation is then studied by the formulator and other participants in the mathematical culture of the day, leading to modification and refinement of the original idea. Their collective efforts shape the mathematical knowledge of their time and set the tone of the mathematical culture for the future. The process of original formulation can have inexplicable and perhaps intuitive aspects that nobody fully understands. On the other hand, once the formulation becomes the common knowledge of the community, many other persons are often stimulated by it, leading to a communal attack on the problem. In this process, certain empirical elements, such as examples of solutions in special cases and counter-examples to tentative hypotheses, etc., can be influential.

Often the kind of question that is originally asked gets mutated to a newer question. Sometimes, a series of mutations of this type leads to new and interesting mathematics that differs sufficiently from the original question to warrant being called by a different name.[91] In another vein, when a particular theorem is "in the air” more than one person can and often does, discover proofs of it. Often only one discoverer is

91 The paradigm is Darwinian, and I believe justifiably so.

remembered. For such events a necessary condition is that there is a community of mathematicians that share a common mathematical culture of that era.[92]

Concerning the transmission of mathematical ideas between different culture-areas, there has been quite a lot of work by several scholars attempting to comprehend the details of the process by which such ideas might have migrated from one culture to another. The papers by Pingree (1973, 1978) and in Kennedy (1983) can give the reader an idea of the types of issues that arise. One can come to certain general conclusions. Namely, certain mathematical facts of a fundamental, but elementary, nature can be and are discovered independently by many cultures. Others follow a pattern of communication followed by extension.[93] In general, the subject of transmission directions and mechanisms is a complex one, fraught with possibilities for error, due to the paucity of the record and the difficulty of comprehending the texture of social life in a historical era from the written record.

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92 In a shared culture, many persons may be contemplating a particular question in an epoch, and any one of them might make an important discovery at any time. Which particular one happens to make the discovery, to which his or her name might (or might not!) be attached, can be a matter of chance. Moreover, subsequent independent discoveries of the same theorem will not generally get the credit they deserve if the earlier version is well-known.

93 More precisely, I believe that in antiquity very often it *was a specific application* of a particular mathematical technique (perhaps conveyed in the form of just a recipe, *with no further conceptual justification*, for solving some problem of mensuration or geodesy) that might have been communicated from one culture to another. In turn, the acquisition of this knowledge might have stimulated the mathematicians of the recipient culture to examine the theoretical basis of the recipe. Such an examination could lead not only to the independent rediscovery of the same mathematical fact by the recipient, but could sometimes lead to a related extension that might not have occurred to the former culture.

8. Miscellaneous notes.

8.1 Transcription and Pronunciation of Arabic, Chinese, and Indian names.

The reader will have noticed that many non-English words or names in the text are accompanied by diacritical marks, which are intended to facilitate the correct pronunciation of those names or words. In this section, I have appended a short guide to the phonetic implications of the marks. I do not aim to be complete.

For diacritical marks in Arabic and Indian (Sanskrit) words, I follow the practice approved by the Library of Congress (see Barry, 1991; see also Rosenthal, 1975 for a brief guide to Arabic transliteration and pronunciation). For Chinese names, matters are more complicated. For the most part, I have used the names as they are given in Needham (1959), who used there the old Wade-Giles system of transliteration. The present day Pinyin system is used in a few names, not cited from Needham. Perhaps I should have used the modern version throughout and transcribed the names as given in Needham to their modern equivalents. This would result in forms that no longer match the source and might create some difficulties for the reader who wishes to consult Needham for additional information on some point related to the names cited here. In any case, the problem of even approximately correct pronunciation of Chinese names is unlikely to be solved by any method of transliteration. I have therefore not tried to include a guide to the pronunciation of Chinese transliterations.

Transliteration and Pronunciation Guide

Transliteration and pronunciation of Arabic words

Consonants

b	Pronounced as in the English language
t	Palato-dontal t (softer than the English t), like t in the French word tout
th	Pronounced as in the English language
j	Pronounced as in the English language
h	Compressed h; this is not found in European languages. (approximately as in hare, but with a glottal constriction)
kh	guttural as in the Scottish loch, or the German nach
d	Pronounced as in the English language
dh	like th in there
r	lingual r
z	Pronounced as in the English language
s	sharp sibilant as in sun
sh	like sh in shout
s	emphatic sharp palatal s
d	emphatic dull palatal d
t	emphatic dull palatal t, somewhat like t in cat
z	emphatic dull palatal z
‘	This mark stands for the Arabic letter "ayn". It is usually denoted by a mark I do not have available to me, so I have used ‘ instead. It is pronounced as an explosive guttural sound, somewhat like an unaspirated counterpart of h. It can only be demonstrated rather than described. It has no English equivalent or even an approximate analog.
gh	rattling guttural between gh and r
f	Pronounced as in the English language
q	dull guttural k
k	Pronounced as in the English language
l	Pronounced as in the English language
m	Pronounced as in the English language
n	Pronounced as in the English language
h	Pronounced as in the English language
’	This mark appearing in a word denotes strong vocal inflection, somewhat like a glottal stop.

Vowels

a	as in material
a	as in father
ay	as in pray
i	as in bin
i	as in pique
aw	as in awful
u	as in pull
u	as in rule

AUTHOR: Gangolli **SUBJECT: Math**

Transliteration and pronunciation of Indian (Sanskrit) words

Consonants

k	like k in seek
kh	aspirated k, as in bulkhead (with the k and h pronounced together)
g	as good
gh	aspirated g as in egghead (with the g and h pronounced together)
n	guttural nasal n, as in angle
c	like ch in beach.
ch	aspirated c as in beachhead (with the ch and h pronounced together)
j	as in joke
jh	aspirated j, (no equivalent in English)
n	palatal nasal n as in anchovy
t	as in cat
th	aspirated t, as in meathook (with the t and h pronounced together)
d	as in dog
dh	aspirated d, as in kindhearted (with the d and h pronounced together)
n	hard palatal n (no equivalent in English)
t	as in French tout
th	aspirated t, somewhat like (but less dontal than) th in thing
d	as in French doux
dh	aspirated d, somewhat like (but less dontal than) in adhere
n	as in no
p	as in pot
ph	aspirated p, somewhat like ph in shepherd, but with h more clearly aspirated
b	as in boy
bh	aspirated b as in subhuman (with the b and h pronounced together)
m	as in man
y	as in yes
r	as in run
l	as in love
l	palatal rolled l, no equivalent in English
v	as in the English v, but more labial (rather than labio-dontal)
s	like sh in shine
s	retroflex palatal sh, no equivalent in English
s	as in sun
h	as in hand

Vowels

a	like the first a in banana
a	as in father
i	as in bin
i	like ee in been
u	as in put
u	like oo in fool
e	like a in lane
ai	somewhat as in nail, but accented
o	as in hole
au	like ou in out

8.2 Chronological table.

This chronology only deals with events or persons that have been mentioned in the preceding sections. It is not intended to be a chronology of the development of mathematics. A negative sign refers to date B.C.; the absence of any sign means that the date is A.D. Please note that the dates are approximate in many cases.

- 1800 to -1600

Babylonian clay tablets on which numerals appear.

-1850

Moscow Papyrus

Contains a method for calculating the volume of a truncated pyramid.

- 1650

Rhind Papyrus

From which most of our knowledge of Egyptian numerals comes.

- 500

Sulvasutras

Writings governing various aspects of ritual. They show an under-standing of the theorem of Pythagoras and deal with many geometric constructions.

- 300

Euclid's *Elements*

- 225

Archimedes

The greatest mathematician of antiquity and one of the greatest mathematicians of all time. He contributed to hydrostatics (principle of buoyancy), mechanics (levers), calculated the surface area and volume of a sphere, made an estimate of π which was not improved until the 17th century.

- 213

Burning of the books in China

- 206 to + 220 (Han dynasty period in China)

Date uncertain

Chiu Chang Suan Shu

(Nine Chapters on the Mathematical Art)

Original version has not survived. A commentary by Liu Hui made about 100 years later has survived. The *Nine Chapters* deals with arithmetic and geometry and has many interesting topics: e.g., Pythagoras' theorem, calculation of π, calculation of the volume of a sphere and of a truncated pyramid, etc. Although the work is ascribed to this period, it could be a compendium of older results in use prior to the Han period.

Date and authorship uncertain

Chou Pei Suan Ching

(The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven)

A work on Astronomy; contains a dissection proof of the theorem Pythagoras.

3rd century
Date uncertain

Sun Tsu Suan Ching
(Master Sun's Arithmetic Manual)

Has many arithmetic problems and recipes for solution. Its numeral system has held sway in China for many years.

Liu Hui

Chinese mathematician. His commentary on the *Nine Chapters* is the principal source of our knowledge of that work. His work contains many results; e.g., an estimate for π, a proof of Pythagoras' theorem, etc.

4th through the 7th centuries

Emergence of the numeral and place value system of the present day.
(Hindu-Arabic numerals with zero.)

5th century

Tsu Chhung-Chih (429-500)

Chinese astronomer. Had a good estimate for π.

Aryabhata (ca. 475-550)

Indian scientist who worked on many branches of science: astronomy, algebra, geometry, trigonometry. Author of the treatise known as *Aryabhatiya.* Using the place value system, he made astronomical calculations dealing with numbers of impressive size and described a comprehensive astronomical system. Worked on Diophantine equations and also the Chinese Remainder problem. Made a trigonometric table of sines, useful in his astronomical calculations.

7th century
622

Muhammad's journey from Mecca to Medina; the birth of Islam.

628

Brahmagupta (ca. 628)

Indian astronomer and mathematician. Author of the *Brahmasphutasiddhanta* an astronomical treatise. In it he had elaborate and quite accurate calculations of planetary periods. He extended Aryabhata's treatise in several ways. Contributed to algebra and arithmetic. Started the study of what is now called Pell's equation, later solved by Bhaskara II.

629

Bhaskara I (Bhaskara the elder) (b. ca. 629)

Indian mathematician. A colleague of Brahmagupta, he worked on Diophantine problems arising from astronomy and wrote a commentary on the work of Aryabhata.

650-750

The period of the Umayyad Caliphs, based in Damascus.

7th to 12th century

Extensive development of calculation procedures in China, India, and West Asia.

750-1258

The period of the ‘Abbasid Caliphs, based in Baghdad.

755-833

The period of the three great Caliphs: al-Mansur, Harun al-Rashid, al-Ma’mun. They were great patrons of learning and the foundations of West Asian scientific scholarship were firmly established in this period.

755

Establishment of the western Caliphate in Spain. The western Caliphate would hold sway at Cordoba till 1492.

9th to 12th centuries

Development of algebra; parallel streams of development in China, India and West Asia; transmission of specific facts not clearly established.

9th century

al-Khwarizmi (d. ca. 860)

Persian mathematician. Author of the *Hisab al-jabr wa-al-muqabala*, the first known systematic work on algebra. The word algebra comes from the word *al-jabr* (meaning restoration or completion) in the title of this work. He was interested in describing step-by-step procedures in solving algebraic equations. Such procedures are now called algorithms, a word derived from his name.

Thabit ibn Qurra (826-901)

Syrian mathematician who was an original scholar as well as a prolific translator. His work contains the earliest known instance of the idea of solving algebraic problems by geometric methods. He translated many Greek works into Arabic (Archimedes, Apollonius, Euclid, Ptolemy, etc.). In particular, he made the first complete translation of Euclid's elements into Arabic.

Mahavira (ca. 850)

Indian mathematician who contributed to the solution of quadratic equations.

10th century

al-Battani (ca. 920)

Arab astronomer who explicated the Ptolemaic system and extended it for his use. His work mentions the law of cosines for a spherical triangle.

Abu al-Wafa’ (940-998)

Persian mathematician who contributed to geometry (parallel postulate) and trigonometry (introduced the tangent function); translator of Diophantus.

11th century

‘Umar al-Khayyam (Omar Khayyam) (1048?-1131?)

Persian mathematician, poet and philosopher. Noted for his work on solution of algebraic problems by geometric methods. Gave a solution of the cubic equation by this method.

12th century

Jayadeva (ca. 1150)

Indian algebraist. He worked on quadratic and Diophantine equations.

Bhaskara II (Bhaskara the younger) (1150-?)

Indian astronomer and algebraist. Worked on many aspects of algebra and astronomy. Noted for his estimate of π; treatise on algebra titled *Lilavati;* solutions of quadratic equations; linear Diophantine equations by means of the method known as the "pulverizer"; used that method to give certain solutions of Pell's equation.

Li Yeh (1192-1279)

Chinese mathematician. His work contains procedures for calculation (multiplication, division, etc.) with rod numerals. In his work one finds a notation for negative numbers (although the concept of negative number was known to Chinese mathematicians long before).

13th century

Nasir al-Din al-Tusi (1201-1274)

Persian mathematician who wrote extensively on geometry. His discussion of Euclid's parallel postulate was translated by Saccheri into Latin and had an impact on European thinking on the axiomatic basis of Euclid's *Elements*.

Yang Hui (ca. 1250)

Chinese mathematician who worked on solutions of geometric problems by algebra. Noticed that certain quadratic equations arising from right triangles can be reduced to linear equations. The first pictorial representation of what is now called Pascal's triangle is found in his work.

Chhin Chiu-Shao (ca. 1201-1260)

Chinese mathematician who wrote a book called *Shu Shu Chiu Chang Zhang* (1247 A.D.). He worked on many topics, including numerical solutions of equations. His method is substantially the same as the method of solution discovered in 1819 by C. Horner. It is now known as Horner's method. See Libbrecht (1973).

14th & 15th centuries

Chu Shi-Kie (ca. 1303)

Chinese mathematician who worked on algebra, counting techniques and numerical computation. His work deals with linear equations in four unknowns, "Pascal's" triangle, and finite differences. See Gillispie (1970).

Ulugh Beg (1393-1449)

A governor of a Persian province. He was of Mongol or central Asian descent and was a learned astronomer. He compiled tables of sines and tangents, accurate to eight decimal places.

8.3 A selection of topics for possible classroom use.

In this brief appendix, I mention a few areas of the mathematics curriculum in which there are possibilities for exposing students to the multi-cultural nature of mathematics. In the case of each curriculum topic, I mention a few possible ways in which the study of how other cultures have approached that topic can be introduced.

A very useful resource for this type of activity is Lumpkin and Strong (1995). See also Williams (1993), which contains many references in which activities suitable for classroom use are described.

Number concept

- Use of different names in other languages, together with a little information about the culture from which the name comes. Learning the different names for a few small numbers can be engaging for children.
- Comparison of different forms and ancestors of our present numerals. Charts of ancestry are fairly easily found from various source books.
- Unusual mechanical counting devices — e.g., Abacus or rod numerals.
- Magic squares. There are well-known and easy algorithms for generating such squares, which are guaranteed to fascinate middle schoolers.
- Calculation of square roots. Heron's method from ancient Greece is easily applied, especially with calculators and is of interest to middle and high school age students. The same method was used by Aryabhata.
- Pythagorean triples and their construction.

Measurement

- A field project in which students actually carry out the method of "double differences" of Liu Hui can be used with beginning high school students.
- The method of Eratosthenes to estimate the circumference of the earth can provide an interesting high school project.

Currency

- Comparisons of different values and exchange rates can provide useful lessons involving ratio and proportion as well as an introduction to different bases for enumeration.

Calendar and time zones

- Planning a trip like "around the world in eighty days" with some constraints (e.g., one must arrive on a certain day at a certain place; it is useful to make this day the same as some festival day in another culture) using actual airline timetables, telephone call rates, etc. could be a very educational activity. There are many other possibilities under this heading.

Geometry

- Generation of various tessellations by rotation around a point followed by rotation about a different point.
- Construction of right triangles by means of knotted chord.
- Simple constructions of regular hexagons, using previously measured, knotted chord.

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