AP CALCULUS AB AND BC

# UNIT 3

# Differentiation: Composite, Implicit, and Inverse Functions

| | | |
|---|---|---|
| AP EXAM WEIGHTING | 9–13% AB | 4–7% BC |
| CLASS PERIODS | ~10–11 AB | ~8–9 BC |

Remember to go to **AP Classroom** to assign students the online **Personal Progress Check** for this unit.

Whether assigned as homework or completed in class, the **Personal Progress Check** provides each student with immediate feedback related to this unit's topics and skills.

## Personal Progress Check 3

**Multiple-choice: ~15 questions**
**Free-response: 3 questions (partial/full)**

UNIT 3

| | | |
|---|---|---|
| AP EXAM WEIGHTING | 9–13% AB | 4–7% BC |
| CLASS PERIODS | ~10–11 AB | ~8–9 BC |

# Differentiation: Composite, Implicit, and Inverse Functions

**BIG IDEA 3**
***Analysis of Functions*** **FUN**

- If pressure experienced by a diver is a function of depth and depth is a function of time, how might we find the rate of change in pressure with respect to time?

## Developing Understanding

In this unit, students learn how to differentiate composite functions using the chain rule and apply that understanding to determine derivatives of implicit and inverse functions. Students need to understand that for composite functions, $y$ is a function of $u$ while $u$ is a function of $x$. Leibniz notation for the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, accounts for these relationships. Units analysis can strengthen the connection, as in $\frac{\text{psi}}{\text{min}} = \frac{\text{psi}}{\text{m}} \cdot \frac{\text{m}}{\text{min}}$. Saying, "times the derivative of what's inside," every time we apply the chain rule reminds students to avoid a common error. Mastering the chain rule is essential to success in all future units.

## Building the Mathematical Practices

1.C 1.E 3.G

Identifying composite and implicit functions is a key differentiation skill. Students must recognize functions embedded in functions and be able to decompose composite functions into their "outer" and "inner" component functions. Misapplying the chain rule by forgetting to also differentiate the "inner" function or misidentifying the "inner" function are common errors. Provide sample responses that demonstrate these errors to help students be mindful of them in their own work. Reinforcing the chain rule structure sets the stage for Unit 6, when students learn the inverse of this process.

Students should continue to practice using correct notation and applying procedures accurately. Checking one another's work, reviewing sample responses (with and without errors), and using technology to check calculations develop these skills. Emphasize that taking higher-order derivatives mirrors familiar differentiation processes (i.e., "function is to first derivative as first derivative is to second derivative").

Use questioning techniques such as, "What does this mean?" to help students develop a more solid conceptual understanding of higher-order differentiation.

## Preparing for the AP Exam

Mastery of the chain rule and its applications is essential for success on the AP Exam. The chain rule will be the target of assessment for many questions and a necessary step along the way for others. One common error is not recognizing when the chain rule applies, especially in composite functions such as $\sin^2 x$, $\tan(2x - 1)$, and $e^{x^2}$. In expressions like $\frac{y}{3y^2 - x}$, students must recognize that the chain rule applies to $y$ because $y$ depends on $x$. When multiple rules apply, students may struggle with the order of operations. Offer mixed practice differentiating general functions using select values provided in tables and graphs. Focus on products, quotients, compositions, and inverses of functions, especially those with names other than $f$ and $g$. Connecting graphs, tables, and algebraic reasoning builds understanding of differentiation of inverse functions.

# UNIT AT A GLANCE

| Enduring Understanding | Topic | Suggested Skills | Class Periods<br>~10–11 CLASS PERIODS (AB)<br>~8–9 CLASS PERIODS (BC) |
|---|---|---|---|
| FUN-3 | **3.1 The Chain Rule** | 1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function). | |
| | **3.2 Implicit Differentiation** | 1.E Apply appropriate mathematical rules or procedures, with and without technology. | |
| | **3.3 Differentiating Inverse Functions** | 3.G Confirm that solutions are accurate and appropriate. | |
| | **3.4 Differentiating Inverse Trigonometric Functions** | 1.E Apply appropriate mathematical rules or procedures, with and without technology. | |
| | **3.5 Selecting Procedures for Calculating Derivatives** | 1.C Identify an appropriate mathematical rule or procedure based on the classification of a given expression (e.g., Use the chain rule to find the derivative of a composite function). | |
| | **3.6 Calculating Higher-Order Derivatives** | 1.E Apply appropriate mathematical rules or procedures, with and without technology. | |

AP Go to **AP Classroom** to assign the **Personal Progress Check** for Unit 3. Review the results in class to identify and address any student misunderstandings.

# SAMPLE INSTRUCTIONAL ACTIVITIES

The sample activities on this page are optional and are offered to provide possible ways to incorporate various instructional approaches into the classroom. Teachers do not need to use these activities or instructional approaches and are free to alter or edit them. The examples below were developed in partnership with teachers from the AP community to share ways that they approach teaching some of the topics in this unit. Please refer to the Instructional Approaches section beginning on p. 199 for more examples of activities and strategies.

| Activity | Topic | Sample Activity |
|---|---|---|
| 1 | 3.1 | **Scavenger Hunt**<br>Place a card with a starter question somewhere in the classroom, for example, "Find the derivative of $f(x) = \sin(4x)$." Place another card in the room with the solution to that card plus another question: "Solution: $4\cos(4x)$. Next problem: Find the derivative of $f(x) = (\sin(x))^4$." Continue posting solution cards with new problems until the final card presents a problem whose solution is on the original starter card (note that this solution should be added to the starter card above). |
| 2 | 3.1 | **Work Backward**<br>Provide a chain rule problem with several possible answers. Have students identify which piece of the original problem contributed its derivative to one of the factors in the answer they are looking at. For each possible answer, students should say whether the answer is correct. They should also identify which piece of the composition was differentiated incorrectly or skipped, or state which factor of that particular answer did not come from the original problem. Students often have trouble knowing when to stop differentiating when learning $e^x$ or trigonometric functions, so make sure you include these types of problems. |
| 3 | 3.2 | **Round Table**<br>In groups of four, each student has an identical paper with four different problems on it. Students complete the first problem on their paper and then pass the paper clockwise to another member in their group. That student checks the first problem and then completes the second problem on the paper. Students rotate again and the process continues until each student has their own paper back. |
| 4 | 3.4 | **Quiz-Quiz-Trade**<br>Give students a card containing a question and have them write the answer on the back. Students then circulate around the room and find a partner. One student quizzes the other by showing only the side of the card with the question on it, and then they reverse roles. They swap cards, find a new partner, quiz each other, and the process continues. |

SUGGESTED SKILL

*Implementing Mathematical Processes*

**1.C**

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

# TOPIC 3.1
# The Chain Rule

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-3**

Recognizing opportunities to apply derivative rules can simplify differentiation.

### LEARNING OBJECTIVE

**FUN-3.C**

Calculate derivatives of compositions of differentiable functions.

### ESSENTIAL KNOWLEDGE

**FUN-3.C.1**

The chain rule provides a way to differentiate composite functions.

TOPIC 3.2

# Implicit Differentiation

SUGGESTED SKILL

*Implementing Mathematical Processes*

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

## Required Course Content

### ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

### LEARNING OBJECTIVE

FUN-3.D

Calculate derivatives of implicitly defined functions.

### ESSENTIAL KNOWLEDGE

FUN-3.D.1

The chain rule is the basis for implicit differentiation.

**SUGGESTED SKILL**

*Justification*

**3.G**

Confirm that solutions are accurate and appropriate.

## TOPIC 3.3

# Differentiating Inverse Functions

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-3**

Recognizing opportunities to apply derivative rules can simplify differentiation.

### LEARNING OBJECTIVE

**FUN-3.E**

Calculate derivatives of inverse and inverse trigonometric functions.

### ESSENTIAL KNOWLEDGE

**FUN-3.E.1**

The chain rule and definition of an inverse function can be used to find the derivative of an inverse function, provided the derivative exists.

TOPIC 3.4

# Differentiating Inverse Trigonometric Functions

SUGGESTED SKILL

*Implementing Mathematical Processes*

1.E

Apply appropriate mathematical rules or procedures, with and without technology.

## Required Course Content

### ENDURING UNDERSTANDING

FUN-3

Recognizing opportunities to apply derivative rules can simplify differentiation.

### LEARNING OBJECTIVE

FUN-3.E

Calculate derivatives of inverse and inverse trigonometric functions.

### ESSENTIAL KNOWLEDGE

FUN-3.E.2

The chain rule applied with the definition of an inverse function, or the formula for the derivative of an inverse function, can be used to find the derivatives of inverse trigonometric functions.

SUGGESTED SKILL

*Implementing Mathematical Processes*

**1.C**

Identify an appropriate mathematical rule or procedure based on the classification of a given expression.

AVAILABLE RESOURCE

- Professional Development > **Selecting Procedures for Derivatives**

## TOPIC 3.5

# Selecting Procedures for Calculating Derivatives

This topic is intended to focus on the skill of selecting an appropriate procedure for calculating derivatives. Students should be given opportunities to practice when and how to apply all learning objectives relating to calculating derivatives.

## TOPIC 3.6

# Calculating Higher - Order Derivatives

**SUGGESTED SKILL**

*Implementing Mathematical Processes*

**1.E**

Apply appropriate mathematical rules or procedures, with and without technology.

## Required Course Content

### ENDURING UNDERSTANDING

**FUN-3**

Recognizing opportunities to apply derivative rules can simplify differentiation.

### LEARNING OBJECTIVE

**FUN-3.F**

Determine higher order derivatives of a function.

### ESSENTIAL KNOWLEDGE

**FUN-3.F.1**

Differentiating $f'$ produces the second derivative $f''$, provided the derivative of $f'$ exists; repeating this process produces higher-order derivatives of $f$.

**FUN-3.F.2**

Higher-order derivatives are represented with a variety of notations. For $y = f(x)$, notations for the second derivative include $\frac{d^2y}{dx^2}$, $f''(x)$, and $y''$. Higher-order derivatives can be denoted $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$.