Rational function: Vertex form – Transformation and Inverse

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Leading question

Are these two functions $\frac{x+2}{x-1}$ and $\frac{1}{x}$ in the same family? If so, how do we describe the transformation of the "child" function and find its inverse using the "common" method of "undo"?

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1. **Vertex form**

The "parent" of rational function has the form $f(x) = \frac{1}{x}$,

therefore, its vertex form is $f(x) = \frac{c}{(x+h)} + k$

Given the rational function $y = \frac{ax+m}{bx+n}$ convert it to vertex form.

Step 1. Divide by function by b (coefficient of the denominator) $y = \frac{\frac{a}{b}x+\frac{m}{b}}{x+\frac{n}{b}}$

Step 2. Rewrite the numerator as a multiple of denominator

$$y = \frac{\frac{a}{b}\left(x+\frac{n}{b}\right)+\frac{m}{b}-\frac{an}{b^2}}{x+\frac{n}{b}} = \frac{\frac{a}{b}\left(x+\frac{n}{b}\right)+\frac{mb-an}{b^2}}{x+\frac{n}{b}}$$

Step 3. Split the numerator $y = \frac{a}{b} + \frac{\frac{mb-an}{b^2}}{x+\frac{n}{b}}$

Back to the question, function $\frac{x+2}{x-1}$ can be rewritten as $\frac{x+2}{x-1} = 1 + \frac{3}{x-1}$. It is the child function in the family of rational function

2. **Transformation**

The transformation of any given function in its vertex form

$$f(x) \rightarrow \; af(bx \pm h) \pm k$$

where |*a*| indicate a quantity of vertical dilate, |*b*| horizontal dilate

k indicates quantity of vertical translate, h horizontal translate
a < 0 → horizontal reflection and *b* < 0 vertical reflection

(think of keeping stretching the line will bend it backward)

For the rational function it its vertex form $y = \frac{a}{b} + \frac{\frac{mb-an}{b^2}}{x+\frac{n}{b}}$

From the parent $y = \frac{1}{x}$ it has translated vertically by $\frac{a}{b}$, horizontally by $\frac{n}{b}$ and vertically dilate by $\frac{mb-an}{b^2}$

Example 1

Define the transformation of rational function $y = \frac{x+2}{x}$

We have $y = 1 + \frac{2}{x}$ (move up 1, stretch by 2)

Example 2

Define the transformation of rational function $y = \frac{3x-2}{x+2}$

We have $y = \frac{3(x+2)-8}{x+2} = 3 - \frac{8}{x+2}$

(move up 3, left 2, stretch by 8 and flip over x-axis)

Example 3

Define the transformation of rational function $y = \frac{2x-3}{3x+2}$

We have $y = \frac{\frac{2}{3}x - \frac{3}{3}}{x+\frac{2}{3}} = \frac{\frac{2}{3}\left(x+\frac{2}{3}\right) - \left(\frac{4}{9}+\frac{3}{3}\right)}{x+\frac{2}{3}} = \frac{2}{3} - \frac{\frac{13}{9}}{x+\frac{2}{3}}$

(move up 2/3, left 2/3, stretch by 13/9 and flip over x-axis)

Back to the question, function $\frac{x+2}{x-1}$ can be rewritten as $\frac{x+2}{x-1} = 1 + \frac{3}{x-1}$. It has moved up 1, right 1 and stretch by 3 times

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3. <u>Inverse</u>

Derive from vertex form $f(x) \rightarrow \ af(bx \pm h) \pm k$

where $|a|$ indicate a quantity of vertical dilate, $|b|$ horizontal dilate

k indicates quantity of vertical translate, h horizontal translate
$a < 0$ → horizontal reflection and $b < 0$ vertical reflection

The inverse function can be found by just doing some switching $a \leftrightarrow$ b and $h \leftrightarrow$ k

$$f^{-1}(x) \rightarrow (b^{-1})f^{-1}(a^{-1}(x \pm k)) \pm h$$

where $|a|$ indicate of **horizontal** dilate, $|b|$ **vertical** dilate

k indicates a quantity of **horizontal** translate, h **vertical** translate

$a < 0$ → horizontal reflection and $b < 0$ vertical reflection

Find the inverse function of $y = \frac{ax+m}{bx+n}$

Step 1: Rewrite in its vertex form (See above)

$$y = \frac{ax+m}{bx+n} = \frac{a}{b} + \frac{\frac{mb-an}{b^2}}{x+\frac{n}{b}}$$

Step 2: Solve for x by cancel the term a/b

$$y = \frac{a}{b} + \frac{\frac{mb-an}{b^2}}{x+\frac{n}{b}} \rightarrow y - \frac{a}{b} = \frac{\frac{mb-an}{b^2}}{x+\frac{n}{b}}$$

And apply the property of equal fraction (Switch places) then isolate the x

$$\rightarrow x + \frac{n}{b} = \frac{\frac{mb-an}{b^2}}{y-\frac{n}{b}} \rightarrow x = \frac{\frac{mb-an}{b^2}}{y-\frac{n}{b}} - \frac{n}{b}$$

Step 3 Switching x and y we have $f^{-1}(x) = \frac{\frac{mb-an}{b^2}}{x-\frac{a}{b}} - \frac{n}{b}$

Example 1

Find the inverse of rational function $y = \frac{x+2}{x}$

We have $y = 1 + \frac{2}{x}$

$$y - 1 = \frac{2}{x} \rightarrow \quad x = \frac{2}{y-1}$$

Switching x and y we have $f^{-1}(x) = \frac{2}{x-1}$

Example 2

Find the inverse of rational function $y = \frac{3x-2}{x+2}$

We have $y = \frac{3(x+2)-8}{x+2} = 3 - \frac{8}{x+2}$

$$y = 3 - \frac{8}{x+2} \rightarrow y - 3 = \frac{-8}{x+2} \rightarrow x + 2 = \frac{-8}{y-3}$$

Or $x = \frac{-8}{y-3} - 2$ Switching x and y we have $f^{-1}(x) = \frac{-8}{x-3} - 2$

Example 3

Find the inverse of rational function $y = \frac{2x-3}{3x+2}$

We have $y = \frac{\frac{2}{3}x-\frac{3}{3}}{x+\frac{2}{3}} = \frac{\frac{2}{3}\left(x+\frac{2}{3}\right)-\left(\frac{4}{9}+\frac{3}{3}\right)}{x+\frac{2}{3}} = \frac{2}{3} - \frac{\frac{13}{9}}{x+\frac{2}{3}}$

From $y = \frac{2}{3} - \frac{\frac{13}{9}}{x+\frac{2}{3}} \rightarrow y - \frac{2}{3} = \frac{-\frac{13}{9}}{x+\frac{2}{3}} \rightarrow x + \frac{2}{3} = \frac{-\frac{13}{9}}{y-\frac{2}{3}}$

Or $x = \frac{-\frac{13}{9}}{y-\frac{2}{3}} - \frac{2}{3}$ Switching x and y we have $f^{-1}(x) = \frac{-\frac{13}{9}}{x-\frac{2}{3}} - \frac{2}{3}$

Back to the question, function $\frac{x+2}{x-1}$ can be rewritten as $\frac{x+2}{x-1} = 1 + \frac{3}{x-1}$ and its inverse is

$$f^{-1}(x) = \frac{3}{x-1} + 1$$

Illustration by desmos https://www.desmos.com/calculator/ufwjqrvxjz